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High Speed Calculating Machines

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The Solution of Simultaneous Linear Equations with the Aid of the 602 Calculating Punch

I. Introduction.—The method of solving simultaneous equations described here is the basic elimination method with certain modifications and improvements. Specifically, the usual elimination method requires the use of a "back-substitution" procedure to evaluate the remaining $(n - 1)$ unknowns after the nth unknown is determined. This "back-substitution" procedure is a departure from the basic elimination formula and requires a change in the elimination technique. In the proposed method "back-substitution" is eliminated, and the same basic elimination procedure is repeated throughout. After n successive reductions, one obtains the values of the n unknowns on n corresponding punched cards.

This improvement is effected by the use of an augmented matrix of somewhat greater dimensions. The character of the particular augmented matrix is determined by the desired end result. In certain cases it is necessary to obtain only the values of the respective unknowns. Other physical problems are such that the matrix of the coefficients of the unknowns remains invariant, whereas the matrix of the constant terms is varied and several solutions are required for the given coefficient matrix. Finally, in some cases it is desirable to compute the inverse of the given matrix of the coefficients of the unknowns. The corresponding variations in the composite matrix for each of these three cases is given in the following mathematical discussion.

The choice of this modified elimination method as the basic scheme to be mechanized was further dictated by its *direct* nature. In contrast to the *indirect* iteration procedures, only one application of the method is necessary to obtain the desired solutions with an accuracy consistent with the number of significant figures in the given matrices and the associated round-off error incurred. There is never a question of the lack of convergence of the method or even the slowness of convergence of the method.

Much has been written regarding the magnitude of the round-off errors incurred in such an elimination procedure.[1,2,3] Since these derivations of the magnitude of the possible round-off errors are based on maximum value criteria, they are very pessimistic. Fortunately, in most practical cases encountered, this error is small. The retention of one, or at the most two, extra guard figures is frequently an adequate precaution against the accumulation of round-off errors.[4]

II. Mathematical Discussion.—Consider the set of simultaneous linear equations,

$$\begin{array}{ccccccc}
a_{11}x_1 & + & a_{12}x_2 & + \cdots & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + \cdots & a_{2n}x_n & = & b_2 \\
\cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot \\
a_{n1}x_1 & + & a_{n2}x_2 & + \cdots & a_{nn}x_n & = & b_n,
\end{array}$$

which may be rewritten in matrix notation as

$$AX = B,$$

in which A is a matrix of order $(n \times n)$ and X and B are single column matrices representing vectors of the nth order.

In the usual elimination method, the equations are so arranged that a_{11} is not zero and the first equation is divided by a_{11}. By successively subtracting a_{k1} (where a_{k1} represent the leading column coefficients of the last $(n - 1)$ equations) times this new equation from each of succeeding $(n - 1)$ equations, x_1 is eliminated from these $(n - 1)$ equations. If the resulting $(n - 1)$ square matrix is similarly treated, x_2 will be eliminated from the last $(n - 2)$ equations. After $(n - 1)$ such reductions, we have a single equation relating the last unknown and a constant term, i.e.,

$$h_{nn}x_n = c_n,$$

which may be solved for x_n. This value of x_n may be substituted back into the next-to-the-last equation relating x_{n-1} and x_n, and the value of x_{n-1} obtained. In a similar manner the remaining unknowns, $x_{n-2}, x_{n-3}, \cdots x_2, x_1$, may be evaluated by "back-substitution."

This "back-substitution" procedure may be avoided by the use of the following composite matrix:

$$(1)\qquad \left\|\begin{array}{c|ccccc|c} \uparrow & a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_{1,n+1} \\ & a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_{2,n+1} \\ & \cdot & & & & & \cdot \\ n & \cdot & & & & & \cdot \\ & \cdot & & & & & \cdot \\ & \cdot & & & & & \cdot \\ \downarrow & a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_{n,n+1} \\ \hline \uparrow & -1 & 0 & 0 & \cdots & 0 & 0 \\ & 0 & -1 & 0 & \cdots & 0 & 0 \\ & \cdot & & & & & \cdot \\ & \cdot & & & & & \cdot \\ n & \cdot & & & & & \cdot \\ & \cdot & & & & & \cdot \\ \downarrow & 0 & 0 & 0 & \cdots & -1 & 0 \\ & \longleftarrow & & n & & \longrightarrow & \end{array}\right\|,$$

which may be rewritten in matrix notation as:

$$\begin{array}{c|c} A & B \\ \hline -I & O \end{array}.$$

If the same basic forward elimination operations are applied to this composite matrix for n successive reductions (see later discussion of Machine Computation for exact details of these operations), we obtain a matrix of

the form:

$$
(2)\qquad \left\|\begin{array}{ccccc|c}
1 & h_{12} & h_{13} & \cdots & h_{1n} & h_{1,n+1} \\
0 & 1 & h_{23} & \cdots & h_{2n} & h_{2,n+1} \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
0 & 0 & 0 & \cdots & 1 & h_{n,n+1} \\
\hline
0 & 0 & 0 & \cdots & 0 & x_1 \\
0 & 0 & 0 & \cdots & 0 & x_2 \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
0 & 0 & 0 & \cdots & 0 & x_n
\end{array}\right\|,
$$

in which the desired n unknowns appear in column $(n + 1)$ *without* the use of the "back-substitution" procedure.

A further consideration of the matrices of (1) and (2) will reveal that certain terms of the negative identity matrix $-I$ contribute nothing during the above elimination process. These terms are those below the main diagonal. Therefore, the "unit upper triangular matrix" may be used with an economy of space and an increase in speed of operation.

In certain physical applications sets of simultaneous equations occur in which the coefficient matrix A is a constant but the constant term matrix B assumes different values for each set of equations. Clearly, such a system of equations is directly amenable to solution. In fact, the entire group of m sets may be simultaneously evaluated with an identical procedure using the following composite matrix:

$$
\left\|\begin{array}{ccccc|cccc}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_{1,n+1} & b_{1,n+2} & \cdots & b_{1,n+m} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_{2,n+1} & b_{2,n+2} & \cdots & b_{2,n+m} \\
\cdot & & & & & \cdot & & & \\
\cdot & & & & & \cdot & & & \\
\cdot & & & & & \cdot & & & \\
\cdot & & & & & \cdot & & & \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_{n,n+1} & b_{n,n+2} & \cdots & b_{n,n+m} \\
\hline
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & & & & \cdot & \cdot & & & \\
 & & & & \cdot & \cdot & & & \\
 & & & & \cdot & \cdot & & & \\
 & & & & \cdot & \cdot & & & \\
 & & & & -1 & 0 & 0 & \cdots & 0
\end{array}\right\|.
$$

This matrix resembles (1) except that additional columns have been added to both the B and O matrices. The reduction procedure is identical with that described above except that $2n(m - 1)$ additional cards are needed.

The evaluation of the inverse of a matrix may be readily accomplished by this method. The decision as to whether a direct solution is to be obtained or whether the inverse matrix is to be evaluated depends upon the physical problem represented as well as the relative magnitudes of m and n.

The inverse matrix provides an accurate indication of the sensitivity of the solution to small changes in the coefficients of A and B. In addition, the unknowns may be readily evaluated for various values of B by means of the following equation:

$$X = A^{-1}B.$$

The actual number of multiplications involved in this matrix multiplication is determined by the respective values of m and n. It is sometimes advisable to solve the following composite matrix:

$$(3)\qquad \left\| \begin{array}{c|c|c} A & B & I \\ \hline -I & O & O \end{array} \right\|.$$

If the same basic elimination procedure is followed, after n reductions the values of the unknowns will appear in the column of the original O matrix located under the B matrix. Furthermore, the inverse matrix A^{-1} will appear in the original $(n \times n)O$ matrix located under the I matrix.

The B matrix and its associated single-column O matrix may be omitted from (3), and only the inverse of A will be obtained. However, the inclusion of the B matrix entails very little additional work, and it does provide an immediate check on the accuracy obtained. That is, the values of the unknowns for one B are available, and they may be substituted into the original equations, and the residuals ascertained.

Sometimes a set of simultaneous equations is obtained with a single column entry for B, it being known however that many *different* values of B will arise in the course of the experimental investigation for a constant value of A. In such a case it is advisable to evaluate (3) in the form given and use the value of the inverse of A to evaluate the unknowns for each new value of B as it occurs in the later investigations.

III. Machine Computation with the Aid of the 602 Calculating Punch. —At any stage of the reduction of the composite matrix

$$(4)\qquad G = \left\| \begin{array}{c|c} A_{nn} & B_{n1} \\ \hline -I_{nn} & O_{n1} \end{array} \right\|,$$

only $(n + 1)$ rows of the matrix are used because of the character of the $-I$ matrix. That is, the leading column coefficient of the $(n + 2)^{\text{th}}$ row is already zero. At each succeeding reduction, one leading row of the composite matrix is omitted and a new row is added—maintaining $(n + 1)$ rows throughout the solution. The number of columns to the left of the vertical line is reduced by one at each reduction. Finally, after the nth reduction, nothing remains to the left of the vertical line, and the desired unknowns appear in the lower right-hand corner.

Punched-Card Layout.—In addition to the actual coefficients which appear in the above matrix some identification of data is necessary to permit efficient machine operation. The following punched-card layout is used:

Card Columns	Data
1–2	row number of element in G
3–4	column number of element in G
5	1 for elements of A
	2 for elements of $-I$
	3 for elements of B
	4 for elements of O
	5 for elements of I
	6 for elements of O^+
6–7	reduction number
8, 9–10	problem number
11–20	sign and data ($P = a_{ij}$) (The decimal point appears between columns 13–14 when seven-decimal places are used in the computation.)
21–30	value of $Q = a_{i1}$ (sign in column 22)
31–40	value of $R = (a_{1i}/a_{11})$ (sign in column 31)
44–53	value of $\|a_{ij} - a_{i1}(a_{1j}/a_{11})\| = \|P - QR\|$.

Key Punching and Verification Procedure.—The respective coefficients of the A and B matrices are supplied to the Key Punch operator on a special coefficient form sheet. In addition to the actual columns of coefficients there is a check column supplied, each entry of which is the sum of all the elements in its corresponding row. Similarly, a check row is supplied, each entry of which in turn is a sum of all the elements in its corresponding column. These check vectors serve two purposes: first, as a check of the key-punched data; second, as a continuous check throughout the actual computation.

The verification of the key punching is effected by means of a suitably wired tabulator board. Each coefficient of the composite matrix is entered on a separate punched card in card columns 11 through 20. The tabulator board is wired to print the successive coefficients of a particular row on the same horizontal line in a tabular array in accordance with the columns of the given matrix. Furthermore, the board is wired to obtain the totals of the columns. It is evident that one can automatically check the key punching of all entries by passing the cards through the tabulator *twice*—once with the cards arranged in order of matrix row and next with the cards arranged in order of matrix column. Any discrepancy between the known check sums will affect two check sums and immediately identify the entry which is in error.

Flow Chart for the 602 Calculating Punch.—The basic elimination process consists of evaluating the following mathematical expression:

$$\|P - QR\| = \|a_{ij} - a_{i1}(a_{1j}/a_{11})\|,$$

in which the use of subscripts will be avoided hereafter by use of the letters P, Q, and R. Because of the repetitive nature of this reduction process, the quantity $(P - QR)$ of one reduction becomes the P of the next reduction. The actual manner in which this is effected may be seen from Figure 1.

Division Cards.—The coefficients of the leading row of the matrix are used to evaluate R in accordance with the equation:

$$R = a_{1j}/a_{11} = D/E.$$

Customer: Yerzuh Prob. No. 1 Ed No. 1 Date 11/29/48

Discussion: Solution of Simultaneous Equations $\left[a_{ij} - a_{i1}\frac{a_{1j}}{a_{11}}\right] = \left[P - Q\frac{D}{E}\right] = \left[P - QR\right]$

	ICAND	IER	LHC		RHC		Summary		Result Storage		
	Divisor 3-4	Quot. 1-2	Dividend 9-12	10-11	5-8	6-7	13-16	14-15			
			RO				RO				
Read Divisor x71	Add E		Add D					Add Iden.			
Prog CTL 3	Sign CTL										
Prog CTL 4			Sign CTL								
Divide	RC	$R = \frac{D}{E}$									
TRANSFER TO STORAGE		RO-RC	RC				Add R				Punch R
Read Detail x68	Add Q		Add - Subt P					Add Iden.			
Prog CTL 1	Sign CTL	Add-Subt R sign CTL			Subt - Add E/2	Provide E/2	Bal CTL RC				
Negative Multiply	RC	RC	(P - QR)		RC						
TRANSFER TO STORAGE			RO				RO				
									(P - QR)		R
Read BLANK CARD								Add 1			
TRANSFER TO STORAGE	RC		RO-RC		RC			RO-RC	(P - QR)	Identification	

FIG. 1. FLOW CHART DIAGRAM OF THE 602 CALCULATING PUNCH.

The punched cards corresponding to the leading row are characteristically designated by an X71 punch. The values of D and E are added into the dividend and divisor counters of the 602, respectively. Since these quantities may be either positive or negative, the algebraic sign must be examined, and the correct sign of the quotient R is determined automatically by the algebraic sign control circuits of the 602 calculator. The quotient is punched on the card and is stored within the calculator for use in succeeding computations.

Multiplication Cards.—The remaining cards of the matrix (exclusive of the leading row X71 cards) are designated by an X68 punch. Each of these cards contains its corresponding matrix coefficient, $P = a_{ij}$, as well as the associated element in the leading *column*, i.e., $Q = a_{i1}$. Since the entire set of cards corresponding to the composite matrix are sorted in order of the

matrix column in which they appear, all X68 cards of a particular column follow the appropriate X71 card.

The values of P and Q are read into the multiplicand and LHC (left hand component) counter, respectively, for each X68 card. The calculator uses the previously stored value of R as a multiplier and performs a combined crossfooting and negative multiplication operation to obtain $P - QR$ with due regard to the algebraic sign of the respective variables. Notice that R is used as a group multiplier for all cards of a particular column of the matrix.

The quantities $(P - QR)$ and R are punched on each X68 card. The value of R is required whenever the product QR is verified by the calculator in a verification-of-multiplication operation. The identification in card columns 1–10 is read into counters 14–15 for both X68 and X71 cards and is retained there until it is transferred to the appropriate columns of the following *blank* card.

Blank Trailer Cards.—Each of the X71 and X68 cards is followed by a blank trailer card. These blank cards receive their appropriate identification by reading out the contents of counters 14–15 into card columns 1–10. They also have the value of $P - QR$ punched upon them in card columns 11–20, i.e., the quantity $(P - QR)$ of reduction no. 1 becomes the new P for reduction no. 2, etc. In effect the 602 performs an interspersed offset-gang-punch operation. Furthermore, the blank cards have the corrected reduction number punched in card columns 6–7. The set of "blank" cards of one reduction becomes the *new* active set in the next reduction.

Operational Steps in a Reduction Process.—The following reduction procedure is equally applicable whether the operation is:

a. the evaluation of the unknowns for a single system of equations,
b. the concurrent evaluation of the unknowns for several sets of simultaneous equations,
c. the evaluation of the unknowns for a constant A matrix and a multiple column B matrix system of equations,
d. the evaluation of the inverse of a matrix, A^{-1}.

The only difference is the respective constituents of the original composite matrix. Specifically, the matrix (4) is used in the solution of n simultaneous equations. The procedure is as follows:[5]

1. *Select* the first $(n + 1)$ rows of the above composite matrix (i.e., all the rows of A and B plus the first row of $-I$ and O). *Sort* on card columns 2, 1.
2. *Sort* on card columns 4, 3, and remove the *leading* column cards from the above selected group.
3. *Gang-punch* X71, X77, and X80 on the first card of this leading column group. *Gang-punch* X68, X77, and X80 on the remaining cards of this leading column group.
4. *Reproduce* a copy of these leading column cards on yellow-top cards EXCEPT for the X77 and X80.
5. *Assemble* the cards in the following order:
   a) X80 cards of the leading column group,
   b) NX80 cards (i.e., cards which do not have an X in position 80) of the leading column group,
   c) remnants of step 2.
6. *Sort* the assembled group in order of increasing matrix row number, i.e., card columns 2, 1.

7. *Offset-gang-punch* "Q" from the leading column X80 cards upon the other cards of the matrix in the SAME row. (X68 and X71 are gang-punched directly).
8. *Verify* the offset-gang-punching.
9. *Sort* on column 80, remove, and file the X80 cards.
10. *Sort* on card columns 4, 3, and arrange the cards in order of matrix column.
11. *Insert* a blank trailer card after each card of this group.
12. *Card count* this merged set, and check the card count against the supplied check data sheet.
13. *Precede* this *new* group of cards with a blank card, and feed the entire group to the 602.
14. *Sort* the cards obtained from the 602 on card column 7, and segregate the former "blank" cards from the original composite matrix. (Label and file the latter.)
15. *Sort* on card columns 4, 3, and remove and file the leading *column* cards of the new matrix (i.e., the former blank cards). These are all equal to zero.
16. *Check* the computation in this reduction by feeding these cards to the tabulator. The *sum* of the entries in a particular row is automatically obtained by the tabulator, and it should agree with the number on the card of the final *check* column.
17. *Sort* on card column 2, 1, and remove and file the leading *row* cards of the new matrix. Again, these should all be zero.
18. *Add* the $(n + 2)^{\text{th}}$ row of the original composite matrix to the *new* matrix (i.e., add the second row of $-I$ and O to the new matrix). Check the card count sheet.
19. Repeat steps 2 to 18 n times.

The cyclic nature of this elimination procedure is clearly shown in Figure 2. The numbers next to the machine diagram refer to the corresponding step in the above reduction procedure. Each reduction is effected by a single passage of the cards around the circular path 2 to 18.

Checking Procedures.—Several of the operations shown in Figure 2 serve only as a check on the reduction procedure. These checks include:

1. an automatic verification of the offset-gang-punching,
2. a continuous check on card count,
3. a verification of the 602 computation of $S = (P - QR)$ by automatically checking the following equation:
$$S - (P - QR) = 0$$
for a zero balance,
4. a computational check by means of the inclusion of *check row* and *check column* cards.

An additional final check may be performed, although it is not shown on the flow chart diagram. This check consists of an evaluation of the residuals which are obtained when the answers are substituted back into the original equations. This check provides an accurate evaluation of the magnitude of the round-off error which is incurred during the elimination process.

Evaluation of the Residuals.—The n cards which are obtained as a result of the nth reduction have the values of the n unknowns in card columns 11–20. If this set of cards is reproduced *reversing* the row and column identification, we obtain x_1 in column 1, x_2 in column 2, etc. If an X71, X22, and a 1 in column 23 are gang-punched on each of these cards, they may be used to evaluate the residuals with the *same* 602 and tabulator boards.

These cards served as "divisor" cards in which the unknown is divided by -1 to obtain the necessary sign reversal in the expression $(P - QR)$. In this case $P = 0$ and R represents the unknowns x_n. The variable Q represents the given coefficients a_{ij}, i.e., in effect we solve the equation

$$(P - QR) = 0 - a_{ij}(-x_n) = a_{ij}x_n.$$

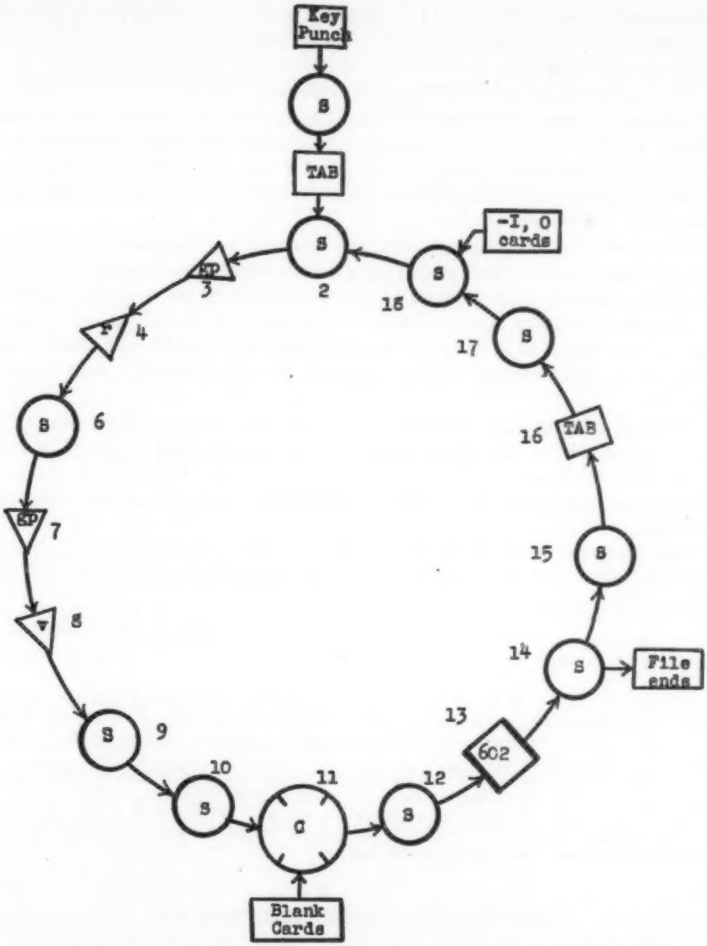

FIG. 2. A MACHINE FLOW CHART DIAGRAM OF THE ELIMINATION METHOD.

Hence, by sorting the original cards of matrix A behind the associated x_n card in order of matrix column, the 602 utilizes the particular x_n as a group multiplier and provides the desired product $a_{ij}x_n$ on each card. If these cards are now fed to the tabulator which totals the cards on matrix row number, we obtain a sum which is the desired residual.

Accuracy Considerations.—The final results are obviously in error by an amount which is determined by the rounding-off errors. One or two guard figures are required in the original coefficient to insure the desired accuracy

in the final answers. The exact magnitude of the error incurred in the case of a set of ill-conditioned equations is still subject to question and requires additional investigation with the aid of high-speed digital equipment.

The usual methods for shifting the decimal point of the given matrix so that the maximum coefficient is near unity are easily effected in this procedure. Furthermore, if it is known (from some physical considerations) that a certain unknown is many times smaller than the rest of the unknowns, its decimal point may be shifted by using -10 or -100 in the corresponding row of the $-I$ matrix.

Obviously the "pivotal condensation" method may be easily incorporated into the described method, since it is a simple matter to select the row with the largest leading coefficient and use it as a pivot in the reduction process.

Computation Time Considerations.—The solution of a single set of equations of the form $(10 \times 10 \times 1)$ requires *four* hours when performed separately. It is quite evident that a multiple set of equations would require less time.

A set of 10 equations with 10 unknowns and multiple right hand sides, e.g., a set involving the same A matrix but 14 columns in the B matrix, $(10 \times 10 \times 14)$ requires ten hours, when a single set is computed. Again it is true that concurrent reduction of several sets will reduce the computation time.

The inversion of a single matrix of ten equations with ten unknowns requires eight hours. Again this time can be reduced by concurrent inversion of several matrices.

FRANK M. VERZUH

Mass. Inst. Technology

[1] J. VON NEUMANN, & H. H. GOLDSTINE, "Numerical inversion of matrices of high order," Amer. Math. Soc., *Bull.*, v. 53, 1947, p. 1021–1099.

[2] A. M. TURING, "Rounding-off errors in matrix processes," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 286–308.

[3] L. FOX, H. D. HUSKEY, & J. H. WILKINSON, "Notes on the solution of algebraic linear equations," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 149–173.

[4] H. F. MITCHELL, "Inversion of a matrix of order 38," *MTAC*, v. 3, p. 161–166, 1948.

[5] Any person interested in obtaining information about the wiring diagrams used in this procedure should write directly to the author.

RECENT MATHEMATICAL TABLES

628[A].—H. E. MERRITT, *Gear Trains including a Brocot Table of Decimal Equivalents and a Table of Factors of all useful Numbers up to 200 000.* London, Pitman, 1947, viii, 178 p. 13.5 × 21.4 cm. Compare *MTAC*, v. 1, p. 21–23, 66–67, 91–92, 100, 132.

T. I. "The factor table," p. 15–54, contains the factors of the 4032 numbers $<200\,000$, having prime factors not less than 7 and not more than 127, the largest convenient tooth number in a change-gear train. **T. II.** "Brocot table," p. 65–95, 6D; the numerator of the fractions is $\leqslant 99$ and the denominator $\leqslant 100$; there is a column of exact remainders for each division.

Extracts from text

629[A, F].—DRAGOSLAV S. MITRINOVICH, "O Stirling-ovim brojevima" [On Stirling numbers], Skoplje, Yugoslavia, Univerzitet, Filozofski Fakultet, *Prirodno-matematichki oddel, Godishen Zbornik* [Year Book], v. 1, 1948, p. 49–95; Russian resumé, p. 90–92; French resumé, p. 93–95. 15.4 × 22 cm.

$x(x-1)(x-2)\cdots(x-n+1) = S_n^1x + S_n^2x^2 + \cdots + S_n^{n-1}x^{n-1} + S_n^nx^n$ and S_n^m are the integral Stirling numbers of the first kind.

$S_{n+1}^m = S_n^{m-1} - nS_n^m$.

If $(x-1)(x-2)\cdots(x-n) = x^n - \phi_n^1x^{n-1} + \phi_n^2x^{n-2} - \phi_n^3x^{n-3} + \cdots + (-1)^n\phi_n^n$,

$$\phi_n^m = (-1)^m S_{n+1}^{n-m+1}, \quad S_n^m = (-1)^{n-m}\phi_{n-1}^{n-m}.$$

There are five tables:

I. $S_n^{n-1} = -\binom{n}{2}$, $n = 2(1)52$; II. $S_n^{n-2} = \frac{1}{4}\binom{n}{3}(3n-1)$, $n = 3(1)65$;

III. $S_n^{n-3} = -\frac{1}{2}\binom{n}{4}n(n-1)$, $n = 4(1)51$;

IV. $S_n^{n-4} = \frac{1}{48}\binom{n}{5}(15n^3 - 30n^2 + 5n + 2)$, $n = 5(1)50$;

V. $S_n^{n-5} = -\frac{1}{16}\binom{n}{6}n(n-1)(3n^2 - 7n - 2)$, $n = 6(1)51$.

S_n^m for $n = 1(1)9$, $m = 1(1)n - 1$ was given by J. STIRLING (1692–1770) in his *Methodus Differentialis*, London, 1730, p. 11; *The Differential Method*, Engl. transl. by F. HOLLIDAY, London, 1749, p. 10. For $n = 1(1)9$, $m = 1(1)n$, also $n = -4(1) - 1$, $m = 0(1)8$, see E. P. ADAMS, *Smithsonian Mathematical Formulae*. Washington, 1939, p. 159. For $n = 1(1)22$, $m = 1(1)n$, see J. W. L. GLAISHER, *Q. Jn. Math.*, v. 31, 1900, p. 26. For $n = 1(1)12$, $m = 1(1)n$ see H. T. DAVIS, *Tables of the Higher Math. Functions*, v. 2, Bloomington, Ind., 1935, p. 215; also given, 1933 by CHARLES JORDAN. See also *MTAC*, v. 1, p. 330.

R. C. A.

630[A–E].—H. M. SASSENFELD & H. F. A. TSCHUNKO, *Mathematische Tafeln für Mathematiker, Naturwissenschaftler, Ingenieure. Erster Teil: Elementare Funktionen.* Walldorf bei Heidelberg, Fr. Lamadé, 1949, viii, 36 p. stiff cover. 20.7 × 29.5 cm.

CONTENTS: **T. 1** (p. 1): P.P., $d = 0(.1)199$; also reprinted on a moveable card. **T. 2** (p. 2–3): Prime factors and 6D mantissae of logs of the prime numbers in the range $n = 1(1)2000$. **T. 3** (p. 4–13): n^2, n^3, $n^{\frac{1}{2}}$, and $n^{\frac{1}{3}}$, 4D, $1000n^{-1}$ 5D (6S for $n \overline{<} 100$), πn and $\frac{1}{4}\pi n^2$ 5–7S, $n = 1(1)2000$. **T. 4** (p. 14): For $\phi = 0(1°)180°$, 5D or 5S values of arc-length l; chord s; segment height h; l/h; segment area F. **T. 5** (p. 14): Spherical area, 6S, and volume, 7S, for $d = 1(1)200$. **T. 6** (p. 15): Arc lengths [0(0°.1)180°; 5D]. **T. 7** (p. 16–17): x^n, $\pm n = 2(1)15$, and $n = -1$, $x = [1.1(.1)10.9; 5S]$. **T. 8** (p. 18): 4D mantissae and antilogs, $x = 10(1)999$. **T. 9** (p. 19): 4D values log sin x, log cos x, log tan x, log cot x, $x = 0(0°.1)90°$. **T. 10** (p. 20–21): tables involving relations between sexagesimal, centesimal and radian systems of measurement. **T. 11** (p. 22–23): sin x, cos x, and tan x, cot x, for $x = [0(0°.1)10°, (0°.1)90°; 5D, 4D]$, $[0(0^g.1)10^g, (0^g.1)100^g; 5D, 4D]$. **T. 12** (p. 24): ln x, $x = [0(.001)1(.01)$-$10(1)109; 5D]$. **T. 13** (p. 25): $e^{\pm x}$, $x = [0(.01)10.09; 5S]$. **T. 14** (p. 26–28): sin x, cos x, tan x, sinh x, cosh x, tanh x, $x^r = [0(0^r.01)6^r$, mostly 5D]. **T. 15** (p. 29): sin x, cos x, $x^r = [6^r(0^r.01)11^r; 5D]$. **T. 16** (p. 30–31): $\sin^{-1} x$, $\cos^{-1} x$, $x = [0(.001).999; 5D]$, $\tan^{-1} x$, $x = [0(.001)1.1(.01)10.09; 5D]$. **T. 17** (p. 32): 9D and 10D values of log x, ln x, $e^{\pm n/100}$, sin $(n/10)$, cos $(n/10)$, sin $(n/1000)$, cos $(n/1000)$, sin $(n/100)$, cos $(n/100)$, $n = 0(1)99$. Also sin $n°$, cos $n°$, n = 0(1°)90°. **T. 18** (p. 33): $x^n/n!$ $n = 2(1)12$, $x = 1.1(.1)10.9$. **T. 19** (p. 34): Miscellaneous numerical tables.

631[A–E, I, K].—L. J. Comrie, *Chambers' Six-Figure Mathematical Tables. Volume I: Logarithmic Values. Volume II: Natural Values.* Edinburgh and London, W. & R. Chambers, 1948, xxii, 576, xxxvi, 576 p. 17 × 25.4 cm. 42 shillings per v. American edition, offset print: New York, D. Van Nostrand Co., 1949, $10.00 per v. or $17.50 for the two v. The binding and type-pages of the British edition are much the more attractive, and the volumes weigh about three quarters of a pound less.

These volumes will be everywhere welcomed as a very valuable compilation of tables. They cover the field of elementary functions, with some minor excursions into no man's land between the frontier of elementary functions and the domain of higher mathematical functions.

In consideration of the fact that even in this day and age there are many who have no access to calculating machines, the editor has considered it worthwhile to compile two volumes, the first, intended for those who have no access to calculating machines, listing the logarithms of functions, and the second, listing natural values of functions, for those to whom calculating machines are available.

The editor has made a convincing case for the desirable tabulation of functions to six decimal places, and has adhered to this policy with but few exceptions (for example: T. II, III, VIII, in v. I; T. XIII in v. II). The intervals in arguments have been chosen so that most tables are essentially linear, i.e., linear interpolation is adequate for six-place accuracy.

To facilitate interpolation, first differences are given for most (but not all) of the tabulated functions, frequently with appropriate proportional parts; in the case of two tables in II second differences are also given. The first differences have been printed in italics, as a warning to the reader, whenever second differences are not negligible.

To obviate interpolation where interpolation is inherently troublesome auxiliary functions have been tabulated. Thus v. I contains tables of $S = \log \sin x - \log x$; $T = \log \tan x - \log x$ and of the corresponding *Sh* and *Th* functions. V. II contains tables of $\sigma = x \csc x$ and $\tau = x \cot x$ and of the corresponding σh and τh functions. Auxiliary functions are also tabulated in the case of inverse circular and hyperbolic functions. Regarding interpolation in the functions $\csc x$ and $\cot x$ it is important to point out the remarkable formulae on p. xiv of II which express the values of $\cot x$ and $\csc x$ for small x in terms of the cotangents and cosecants of the angles $10x$ and $100x$. An important feature worth mentioning is the inclusion of a very large number of critical tables, i.e., tables in which the functional values appear at equidistant intervals, the corresponding arguments being unevenly spaced.

The texts preceding the tables consist of introductions followed by detailed descriptions of the various tables, including numerical examples illustrating the use of the tables, with particular emphasis on both direct and inverse interpolation. The introductions contain a fairly extensive discussion of such topics as the motivation for the choice of the range and intervals of the tables, the essential facts underlying the theory of direct and inverse interpolation, numerical differentiation and integration, etc. As was to be expected the introductions include a discussion of the throw-back method of modifying differences, popularized by the editor, his technique of inverse interpolation and his technique of applying the Lagrangean interpolation formula in the case of unequal intervals. The bibliographies at the end of I–II contain the most important references to other tables of the functions tabulated in I–II with more than 6D.

No one will challenge the editor's statement that "great attention has been paid to typography"; nevertheless the reviewer feels that the legibility of T. VII in I, would have been enhanced by lines separating the columns and by providing spaces between groups of values corresponding to five or perhaps ten arguments. Also, the reviewer does not favor the use of the headings sin, tan, or sin x and tan x in I where log sin x and log tan x are intended.

While the reviewer agrees with the editor's comments on the shortcomings of interpolation by means of Lagrangean interpolation coefficients, he feels however that interpolation by means of differences is not altogether free from similar shortcomings.

The reviewer has not sampled the volumes for accuracy (freedom from error). There is, however, no doubt in his mind that they come up to the usual high standards which one is accustomed to find in "Comrie" tables.

We shall now indicate some of the essential contents of the volumes; a number of small auxiliary and critical tables have been omitted from this account.

I—T. I (p. 2–181): logarithmic mantissae for 10,000(1)100,009.

T. II (p. 182–191): 8D logarithmic mantissae, and 6D values of Mx and x/M, for $x = 0(.001)1$, together with 6D values of multiples, 1(1)10(10)90, of M and $1/M$, for the purpose of converting from common to natural logarithms and vice-versa.

T. III (p. 192–211): 8D values of logarithms for 1(.00001)1.10009. One use of this table is in compound interest and annuity problems; a second use is in the evaluation of 8D values of logarithms by the factor method in conjunction with T. II.

T. IV (p. 212–231): antilogarithms, giving the values of 10^x for $x = 0(.0001)1$.

The next three tables (p. 232–340) are devoted to the tabulation of the logarithms of trigonometric functions of angles expressed in degrees, minutes and seconds:

T. VA: log sin x and log tan x for $x = 0(1'')1°20'$.

T. VB: log sin x, log cos x, log tan x, and log cot x for $x = 0(10'')10°$.

T. VC: log sin x, log cos x, log tan x, and log cot x for $x = 10°(1')45°$.

The next four tables (p. 341–489) give the values of trigonometric functions of angles expressed in degrees and decimal subdivision of the degree and in radians:

T. VIA: critical table for log cos r where r varies from 0 to $0^r.025$, and for log sin r where r varies from $1^r.546$ to $1^r.571$ ($\frac{1}{2}\pi$).

T. VIB: values of log sin and log tan for angles 0(.001°)5° together with the radian measures of the angles.

T. VIC: rounded off proportional parts for 10(10)170 units of the last decimal of the radian arguments for the differences 2(1)231(3)876.

T. VID: logarithms of sin, cos, tan, and cot for the angles 0(.01°)45° and the corresponding values of the angles in radians.

T. VII (p. 490–499): critical table for the functions $S = \log \sin x - \log x$ and $T = \log \tan x - \log x$ for x expressed in seconds of arc, minutes of arc, degrees and decimal subdivision of the degree, radians, seconds of time and minutes of time. In all cases the values of log sin x and log tan x corresponding to the unevenly spaced arguments x are also given.

T. VIII (p. 500–539): logarithms of the hyperbolic sin x, cos x, and tan x for $x = 0(.001)$-3(.01)5 and several auxiliary tables (critical tables, proportional parts, etc.).

T. IX (p. 540–543): logarithms of $\Gamma(x)$ for $x = 1(.001)2$.

T. X (p. 544–549): conversion of degrees, minutes, and seconds into radians, radians into degrees and decimal subdivision of the degree, radians into degrees, minutes and decimals, radians into degrees, minutes and seconds, and other conversion tables.

T. XI (p. 550–569): first nine multiples of the numbers 1(1)999.

II—There is some "overlapping" between **I** and **II**. Specifically the first three tables in **II** (p. 2–215) are the counterparts of T. V, VI, and VII in **I**, and are devoted to the tabulation of the natural values of the trigonometric functions while T. IV is the counterpart of T. VIII in **I**, and contains a tabulation of the hyperbolic functions. T. IVD, IVE, IVF, IVG of **II** include values of $e^{\pm x}$ for 0(.001)3(.01)6(various) ∞.

T. V (p. 318–335): natural logarithms for $x = 0(.001)10$.

T. VIA (p. 336–355): $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$ for $x = 0(.001)1$.

T. VIB (p. 356–357): $\sin^{-1} x$ and $\tanh^{-1} x$ and some auxiliary functions for x in the range from .99 to 1.

T. VIC (p. 358–359): $\sec^{-1} x$, $\cosh^{-1} x$, $\coth^{-1} x$, and some related functions in the range from 1 to 1.01.

T. VID (p. 360–401): inverse functions which can have values larger than unity for $x = 1(.001)2(.01)10(.1)35(1)80$. The frequent use of italics indicates that the tables are "non-linear," i.e., linear interpolation will not give six-place accuracy.

T. VII (p. 402–425): gudermannian and its inverse; $\mathrm{gd}x = \int_0^x \operatorname{sech} x\, dx$ is tabulated for $x = 0(.001)4.5(.01)10$; and $\mathrm{gd}^{-1}x$ for $x = 0(.001)1.400(.0001)1.57050(.00001)(1.570780$-$(.000001)1.570796$.

T. VIIIA (p. 426–465): $x^2, x^3, x^4, x^5, x^{\pm\frac{1}{2}}, x^{-2}, x^{-1}, x^{1/3}, x^{1/4}, x^{1/5}, x!$, log $x!$ and the prime factors of $x = 1(1)1000$.

T. VIIIB (p. 466–489): x^2, and $(x/1000)^3$ and the prime factors of $x = 1000(1)3400$.

T. IX (p. 490–491): prime numbers up to 12919.

T. X (p. 492–513): to facilitate the conversion from rectangular to polar coordinates, $1 - k^2$, $(1 - k^2)^{\frac{1}{2}}$, $(1 + k^2)^{\frac{1}{2}}$, $\tan^{-1} k$ (both in degrees and radians) and $\cot^{-1} k$ for $k = 0(.001)1$. If s is the smaller of the numbers x and y and l is the larger, then $(x^2 + y^2)^{\frac{1}{2}} = l[1 + (s/l)^2]^{\frac{1}{2}} = l(1 + k^2)^{\frac{1}{2}}$ and $\tan^{-1}(y/x)$ is either $\tan^{-1} k$ or $\cot^{-1} k$. The values of $1 - k^2$ and $(1 - k^2)^{\frac{1}{2}}$ have been included because of their use in certain statistical problems.

T. XI (p. 514–517): $\Gamma(x)$ for $x = 1(.001)2$.

T. XIIA (p. 518–519): erf x for $x = 0(.01)4$.

T. XIIB (p. 520–524): $z = (1/(2\pi)^{\frac{1}{2}})e^{-\frac{1}{2}x^2}$, for $x = 0(.01)5$, and $\alpha = (2/\pi)^{\frac{1}{2}} \int_0^x e^{-\frac{1}{2}t^2}dt$, for $x = 5(.01)6$; also the values of certain related functions.

T. XIIC (p. 525–531): probability abscissa x and ordinate z, the argument $\frac{1}{2}(1 + \alpha)$ being the integral between $-\infty$ and x, for respective arguments .5(.001)1 and .99(.0001)1.

T. XIII (p. 532–543): for the coefficients B'' and B''' in Bessel's interpolation formula, where the B's are functions of n, the fraction of the tabular interval for which the fundamental value is desired. Also two short tables of 4-point and 6-point Lagrangean interpolation coefficients.

T. XIV (p. 544–549): formulae for numerical differentiation and integration.

T. XV (p. 550–561): proportional parts for $6''(1'')10''(10'')50''$ for the differences 2(1)1015.

T. XVI (p. 562–567): conversion table identical with T. X in I.

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632[A–E, K, M].—HAROLD D. LARSEN, *Rinehart Mathematical Tables*. New York, Rinehart & Co., 232 Madison Ave., 1948, viii, 264, [2] p. 14 × 21 cm. $1.50.

PARTIAL CONTENTS: **T. 1–3**, p. 3–94, 5D tables of logs of numbers, trigonometric functions (sin, tan, cot, cos) at interval 1′, log trig. functions. **T. 4**, p. 95–114: $N = 1(1)1000$, N^2, N^3, $N^{\frac{1}{2}}$, $(10N)^{\frac{1}{2}}$, $N^{\frac{1}{3}}$, $(10N)^{\frac{1}{3}}$, $(100N)^{\frac{1}{3}}$, $1000/N$. **T. 5**, p. 115–116: Four-place log. **T. 6**, p. 117–120. 5D Nat. trig. functs., 0(0°.1)90°. **T. 7**, p. 121–124, 5D log trig. functs. at interval 0°.1. **T. 13**, p. 131: 5D log. factorials. **T. 14**, p. 132–135: Four-place ln. **T. 15**, p. 136–141: e^x, e^{-x}, log e^x, $x = 0(.01)5(.05)10$. **T. 16**, p. 142–145: sinh x, cosh x, tanh x, $x = 0(.01)$-$3(.05)7.5(.25)10$. **T. 17–18**, p. 146–147: Mortality tables. **T. 20–23**, p. 150–153: $(1 + r)^{\pm n}$, $\pm [(1 + r)^{\pm n} - 1]/r$. **T. 24–25**, p. 154–155: Ordinates and areas of the normal probability curve. **T. 26**, p. 156–159: Values of F and t 5% and 1% points. **T. 27**, p. 160: Values of χ^2 corresponding to certain chances of exceeding χ^2. Various formulae and constants, p. 164–199. Curves for reference, p. 200–214. Derivatives, p. 215–217. Indefinite integrals (430), p. 218–250. Definite integrals (431–494), p. 251–255. Series, p. 256–260. Index, p. 261–264. Proportional parts, 2 p.

633[A–F, H, L–N].—M. BOLL, *Remarques et Compléments aux Tables Numériques Universelles*. Paris, Dunod, 92 rue Bonaparte (VI), 1949, 32 p. 18.2 × 27 cm. The publisher will supply these pages free to any purchaser of the original work, published in 1947; compare *MTAC*, v. 2, p. 336–338. This pamphlet is dated, inside, 9 Nov., 1948.

It contains a list of corrections and additions to the original work, p. 4–882. Only one of the numerous errors which we listed has been corrected.

As an addition to the original, p. 26–45, a new table of $n^{\frac{1}{2}}$ is given (p. 4–8) for $n = [1(1)1000; 7S]$.

T. 13, p. 62–70: "Nombres premiers et plus petits diviseurs" has been completely discarded in favor of a new table of the same character, p. 10–18.

Various additions including Euler numbers are set forth for p. 232.

A new table of $\tan x - x$, for $x = [1°(1°)60°; 6D]$ is given (p. 21) as a supplement to p. 234.

The old Fresnel integrals tables, p. 362–363, uniformly 4D, are replaced by corrected tables (p. 23–24) with many 5–6D values.

To the regular polyhedra table, p. 374, an addition is given on p. 25.

Many corrections are given (p. 30) for the table of roots of cubic equations p. 710–711.

R. C. A.

634[A, C, D].—HEINZ WITTKE, *Vademekum für Vermessungstechnik*. Stuttgart, Metzlersche Verlagsbuchhandlung, 1948, 334 p. 10.5 × 15.4 cm. See also the author's work on *Rechenmaschine*, *MTAC*, v. 3, p. 390.

This is an elementary miscellany useful for the surveyor. Among the 24 tables, p. 131–334 are the following 5D tables, p. 132–279: $\log x$, $x = 1000(1)10009$, Δ; log sin, log tan, log cot, log cos, for $0(0^g.01)10^g(0^g.1)100^g$, Δ, PP (beginning with 2^g); natural sin, tan, cot, cos, for $0(0^g.1)100^g$, Δ, PP. Then there is also a 4S table of $100 \cos^2 \alpha$ and $50 \sin 2\alpha$ for $\alpha = 0(0^g.1)100^g$, p. 290–299.

R. C. A.

635[C, D].—VEGA, *Seven Place Logarithmic Tables of Numbers and Trigonometrical Functions*. New York, Hafner Publishing Co., 1948, XVI, 575 p. 15 × 22.9 cm. $3.00.

CARL BREMIKER (1804–1877) edited the fortieth edition of the *Logarithmisch-Trigonometrisches Handbuch* of GEORG, FREIHERR VON VEGA (1756–1802), published in Berlin by Weidmann, in 1856, XXXII, 575 p. The interesting preface occupied p. I–XVI, and the Introduction p. XVII–XXXII. In the following year the same publisher issued an English translation, prepared by W. L. F. FISCHER (1814–1890), F.R.S. 1855, with the title: *Logarithmic Tables of Numbers and Trigonometrical Functions by Baron von Vega. Translated from the fortieth or Dr. Bremiker's thoroughly revised and enlarged edition.* Fischer was a fellow of Clare College Cambridge and professor, at the University of St. Andrews, of natural and experimental philosophy (1849–59), and of mathematics (1859–1879). *Alumni Cantabrigienses* tells us that his Christian names were "Frederick William Lewis (or Wilhelm Ferdinand Ludwig)," that he was born in Magdeburg, Prussia and naturalized in 1848, and that he was fourth wrangler in 1845. Of this Fischer edition there were numerous reprints such as the 83rd edition of Vega in 1912. The first edition of Vega's *Handbuch* was in 1793, and the 19th edition, 1839, was edited by J. A. HÜLSSE. The Italian translation of the Vega-Bremiker *Handbuch* by LUIGI CREMONA (1830–1903), and a French translation were also published by Weidmann in 1857, as well as a Russian translation in 1858.

The volume under review, denuded of every name except that of Vega, is simply Fischer's English translation of Bremiker's edition of Vega's tables with the 16 pages of Bremiker's historical preface eliminated. In 1941 G. E. Stechert & Co. (the forerunner of the Hafner Publishing Co.) published a similar English edition, but with the original German preface and introduction.

The first long table (p. 1–185) is $\log N$, $N = 1(1)10009$, P.P., with S and T tables at bottoms of pages.

The second principal table (p. 187–287) is of $\sin x$ and $\tan x$, $x = 0(1'')5°$.

The third main table (p. 289–559) is of $\sin x$, $\tan x$, $\cot x$, $\cos x$, $x = 0(10'')45°$, Δ.

An Appendix (p. 561–575) contains: Tables for the conversion of (i) sidereal time into mean time; (ii) mean time into sidereal time. Tables of refraction. Constants.

See J. HENDERSON, *Bibliotheca Tabularum Mathematicarum*, Cambridge, 1926, p. 126–127.

R. C. A.

636[F].—G. N. WATSON, "A table of Ramanujan's function $\tau(n)$," London Mathematical Soc., *Proc.*, s. 2, v. 51, 1949, p. 1–13; in the press since 1942. Compare *MTAC*, v. 2, p. 26–27; v. 3, p. 23, 177, 298.

This table gives the coefficients $\tau(n)$ of x^{n-1} in the power series expansion of the 24th power of EULER's product $(1-x)(1-x^2)(1-x^3)\cdots$ for $n = 1(1)1000$. As noted above it has already been discussed in this journal. Its appearance will be very welcome to investigators of this mysterious function since it more than doubles the range of previous tables, described in the above reference. Besides the function $\tau(n)$ the table gives $\tau(n)n^{-\frac{11}{2}}$ to 5D. According to a conjecture of RAMANUJAN this function is less than 2 in absolute value when n is a prime. The table shows that this function with n a prime lies between -1.91881 and $+1.90410$ and attains these values for $n = 103$ and 479 respectively. For the composite number 799 the function is equal to -2.01623.

The method used to construct the table is the same as that described and used by the reviewer.[1] When n is composite short methods of calculation yield $\tau(n)$. When n is a prime a comparatively long recurrence formula, based on Euler's pentagonal number theorem, is used. In this latter case the author used the long calculation for both n and $n + 1$ to ensure accuracy.

D. H. L.

[1] D. H. LEHMER, "Ramanujan's function $\tau(n)$," *Duke Math. Jn.*, v. 10, 1943, p. 483–492. See *MTAC*, v. 1, p. 183–184.

637[G, I, K].—WILLIAM EDMUND MILNE, *Numerical Calculus. Approximations, Interpolation, Finite Differences, Numerical Integration, and Curve Fitting*. Princeton, N. J., Princeton Univ. Press, 1949, x, 393 p. Offset print. 15 × 22.8 cm. $3.75.

This introductory treatise on numerical analysis contains one of the finest and most thorough presentations of the subject by a well-known specialist in that field. In the space of a modestly-sized volume the author succeeds in explaining the essentials of finite differences, interpolation, numerical integration and differentiation, important techniques in smoothing and approximations by least squares, and the solution of simultaneous linear equations, algebraic equations, and difference equations. The text is supplemented by an adequate bibliography and useful tables. Also the presentation of the theory is reinforced with clear illustrative examples and sets of exercises at the ends of the chapters. Besides serving well as a standard text for an undergraduate course in finite differences and numerical methods, this book provides more than an adequate foundation in numerical analysis for those who seek to specialize in applied mathematics or branches of science involving calculation, such as statisticians, computers, actuaries, engineers, physicists and biologists. Furthermore, all mathematicians, even those of the purest type, should be cognizant today of the subject matter in this book because of the enormous development of numerical analysis by present-day electronic computational devices, and its probable influence upon mathematics as a whole. The primary purpose of this book is to enable one with a previous background of only high school mathematics, analytic geometry, and elementary calculus, to bridge the gap between classroom mathematics and the important practical applications where concrete numerical results are required, or to paraphrase the author in his preface, "the gap between knowing of a solution and actually obtaining it." The entire work is written simply, clearly and directly; the treatment is elementary. But besides covering most of the fundamentals of numerical calculation, a number of special topics (e.g., treatment of the remainder in quadrature formulae, alternative treatment of smoothing formulae, exceptional cases in reciprocal differences) are treated with an appealing thoroughness which is also capable of stimulating the reader to further deeper investigations. Even the experienced worker in numerical mathematics, who has in his possession the other standard books of MILNE-THOMSON, SCARBOROUGH, STEFFENSEN, JORDAN, WHITTAKER & ROBINSON, etc., could still find it advantageous to add this treatise to his collection, for its value in ready

reference, as a quick refresher, and for its few features that are not in those other texts, but which are otherwise available only in scattered sources.

Chapter I treats determinants, systems of linear equations, matrices, and homogeneous equations. Its most noteworthy feature is its exposition of one of the most convenient methods for solving a system of linear equations, the concise scheme of elimination described by P. D. CROUT, which is suited to the evaluation of determinants and calculating the inverse of a matrix, and is thus adapted to solving a number of different sets of equations having the same left members but different right members. There follows a discussion of the magnitude of the inherent error, the uncertainty in the result due to the initial uncertainty in the values of the coefficients of the unknowns, which cannot be reduced by any improvement in the technique of solution.

Chapter II deals with the solution of non-linear equations in one or more variables by various methods of successive approximations. For one equation in one unknown, there is described Newton's method and some simpler variations (as a fixed slope m in place of $f'(x_n)$ which is employed at the nth step of the iteration to obtain x_{n+1}). Also the exceptional case, where $f'(x)$ vanishes or is small near the root, is explained. For solving two equations in two unknowns there is given both the extension of Newton's method and successive substitutions, together with a detailed study of the exceptional cases where there may be multiple solutions, distinct solutions close together, or no solution. A procedure is given for finding the complex roots of algebraic equations with real coefficients, by synthetic division by quadratic factors, the end result being the real quadratic factor that yields a pair of complex roots. A very simple iterative scheme is described for the solution of symmetrical λ-determinants whose elements a_{ij} correspond to a positive definite quadratic form. The method enables one to find simultaneously with the λ's, the sets of x_j's corresponding to the different roots λ, which satisfy the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = \lambda x_i, \quad i = 1, 2, \cdots, n.$$

Chapter III introduces the notion of interpolating function of polynomial, rational, or trigonometric type, and then is devoted entirely to polynomial interpolation. AITKEN's process for reducing any degree polynomial interpolation to a succession of linear interpolations is explained together with NEVILLE's variation of Aitken's principle. Then, Aitken's method is applied to inverse interpolation. The usual expression for the error in nth degree polynomial approximation in terms of $f^{(n+1)}(x)$ is derived from ROLLE's theorem. The rest of the chapter contains a detailed account of LAGRANGE's interpolation formula and its usefulness for functions that are tabulated at equally spaced intervals when tables of Lagrangian interpolation coefficients can be employed.

Chapter IV, on numerical differentiation and integration, deals mainly with the latter topic, which is the more important. For differentiation (first derivative only) based upon the approximation by Lagrange's polynomial, an estimate for the error is obtained for the derivative at one of the points of interpolation, and differentiation formulae are given up to the seven-point case. Numerical integration is based also upon Lagrange's polynomial and the method of "undetermined coefficients" is described, which merely means the finding of coefficients by solving a system of linear equations arising from the integration (or differentiation) of a very simple set of polynomials like x^n, $n = 0(1)n - 1$. The investigation of the error is interesting and somewhat advanced, but well worth mastering since the final result has wide application. The author considers the remainder of a quadrature formula $R(f)$, as special cases of operators of degree n, when $R(x^m) = 0$ for $m \leqslant n$, but $R(x^{n+1}) \neq 0$. First $R(f)$ is obtained in the form $\int_{-\infty}^{\infty} f^{(n+1)}(s)G(s)ds$, where $G(s) = (n!)^{-1}R_x[\overline{(x - s)^n}]$, with $\overline{(x - s)^n} = (x - s)^n$ if $x > s$, and $\overline{(x - s)^n} = 0$ if $x < s$, so that $G(s)$ can be evaluated. Under a few mild restrictions $|R(f)| < \max|f^{(n+1)}(s)| \cdot \int_{-\infty}^{\infty}|G(s)|ds$. When $G(s)$ does not change sign, $R(f) = f^{(n+1)}(z)R[x^{n+1}/(n + 1)!]$, least $x < z <$ greatest x. The advantage in this latter form is that $R[x^{n+1}/(n + 1)!]$ is easier to evaluate than $\int_{-\infty}^{\infty} G(s)ds$. This theory is then applied to the trapezoidal rule, to several "trapezoidal rules with corrections," to

Simpson's rule, and finally to the series of Newton-Cotes quadrature formulae. Both closed and open types of Newton-Cotes formulae are tabulated as far as the nine-point case and seven-point case respectively, with a discussion of their merits and the remainder terms.

Chapter V is on the numerical solution of differential equations by stepwise methods. A very simple integration formula is used to introduce the subject. A second method employs two formulae, one as a "predictor," and the other as a "corrector." Special formulae are given for second-order differential equations. Simultaneous equations are also touched upon. The short chapter concludes with some useful five-term integration formulae.

Chapter VI introduces the subject of finite differences. The fundamental properties of factorial polynomials and binomial coefficient functions precede the definition and illustration of differences, with an indication of the important role of differences in the detection of errors in tabulated functions. Then NEWTON's interpolation formulae, employing forward and backward differences, are readily derived from the properties of Δ^k and factorial polynomials, which are also employed to give SHEPPARD's rules. GAUSS's forward and backward interpolation formulae are obtained here from Sheppard's rules. Then the definitions of central and mean-central differences are introduced, followed by STIRLING's central difference interpolation formula. From Gauss's forward formula, in a few neat steps, the highly useful and elegant EVERETT central difference interpolation formula is derived. Finally, from Gauss's forward and backward formulae BESSEL's interpolation formula is derived. The tabulation of polynomials is discussed from the standpoint of building them up from differences. The important role played by differences in subtabulation of a function from interval H to the smaller h is shown from the symbolic formula $\Delta_h^m = [(1 + \Delta_H)^{h/H} - 1]^m$. Formulae for derivatives in terms of differences are found by differentiation of Stirling's and Newton's interpolation formulae. By integrating Newton's interpolation formula, the author derives both LAPLACE's formula for numerical integration over a single interval, and several more general formulae for integration over a number of intervals, the most noteworthy being GREGORY's formula where the differences involve only ordinates in the range of integration. Integration of Stirling's formula is employed to yield a number of elegant formulae for $\int_{x_{-n}}^{x_n} f(x)dx$ (symmetric integrals), in terms of f_0 and $\delta^{2m}f_0$ as far as $\delta^{10}f_0$, for $n = 1(1)5$. Finally, Everett's formula is integrated to give the rapidly convergent GAUSS-ENCKE formula, whose differences use ordinates outside the range of integration.

Chapter VII is devoted to divided differences, their definition, proof of symmetry in their arguments, and the derivation of the fundamental Newton divided difference formula for polynomial interpolation. Then the flexibility of divided differences in giving a number of different versions of the same interpolating polynomial is shown by means of Sheppard's rule.

Chapter VIII treats reciprocal differences and the approximation of functions by rational fractions. The general idea of the reciprocal difference is arrived at from the determinantal form of the interpolating rational fraction. THIELE's definition of reciprocal differences and his interpolating continued fraction are discussed with great thoroughness. The author shows both symmetry, and the expression of reciprocal differences as a quotient of two determinants. Then, after defining the "order" of a rational fraction, a uniqueness theorem is proven for interpolation by a rational function of given order k for $k + 1$ points. Also there is proved the constancy of the kth order reciprocal differences for irreducible rational fractions of order k, and the converse. The chapter ends with a detailed study of exceptional cases and a sufficient condition for the existence of a unique irreducible fraction of order k, which is determined by $k + 1$ points.

Chapter IX deals with the important subject of polynomial approximations by least squares. The normal equations are derived for the unknown coefficients as a necessary criterion for minimizing the sum of the squares of the errors, and with the aid of determinants whose elements are $s_k \equiv \Sigma x_i^k$, these equations are shown always to possess a unique solution. Furthermore, a sufficient condition for this solution to give a minimum, is shown to hold always. When the integral of the squares of the errors, instead of the sum, is to be minimized, the approximating polynomial is given as a sum of LEGENDRE polynomials. The explicit expression for the mth Legendre polynomial $P_m(x)$ is derived from the orthogonality

condition $\int_0^1 x^s P_m(x)dx$, $s = 0(1)m - 1$. $P_m(x)$ is given explicitly as far as $m = 8$; it is shown that $\int_0^1 P_m^2(x)dx = (2m + 1)^{-1}$; and the roots of $P_m(x)$ are shown to be real, distinct, and between 0 and 1. Since the method of least squares via the solution of the normal equations becomes too laborious beyond the third or fourth degree approximation, a more convenient alternative method utilizes sets of polynomials which possess orthogonality properties relative to sums, entirely analogous to the Legendre polynomials for integrals (first considered by CHEBYSHEV in 1858), and the author develops the subject in a manner similar to that for the Legendre polynomials. The subject of graduation or smoothing of data is treated first from the least squares viewpoint, employing these orthogonal polynomials under the assumptions that the observations of the unknown function $f(x)$ are equally spaced in x, and that $f(x)$ can be represented with sufficient accuracy by a polynomial of degree m over several consecutive values. Explicit formulae are given for $m = 3$ and 5, for 7(2)21 points of observation. An alternative treatment of smoothing formulae, under four reasonable assumptions involving the random errors, leads to those same formulae, but in terms of central differences, together with an estimate of the improvement in the smoothing formula over the use of unsmoothed entries. Gauss's quadrature formula using the zeros of the Legendre polynomials is derived, and the zeros with corresponding weight factors are tabulated for $P_m(x)$, $m = 2(1)9$, to 7D.

Chapter **X** considers other approximations by least squares, not necessarily using polynomials, for both the continuous and discrete cases. After introducing the general problem of least squares approximation, and the role of orthogonal functions, a fundamental theorem is proved which shows that when the "true function" u is assumed to be a sum of orthogonal functions, a function y, obtained as an approximation to an observed function z, is closer to u than z itself. Trigonometric interpolation is developed first from the standpoint of the FOURIER series, whose coefficients minimize the integral of the square of the difference, and then from the standpoint of harmonic analysis whose coefficients (resembling Fourier coefficients, but in terms of sums instead of integrals) minimize the sum of the squares of the difference. A method for computing the coefficients in harmonic analysis is described. The GRAM-CHARLIER approximation for functions approaching zero very rapidly as $|x|$ becomes infinite leads, in the continuous case, to the HERMITE polynomials $H_m(x)$ and the calculation of the coefficients c_i of $f(x)$ as a sum $(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2} \sum_{i=0}^{n} c_i H_i(x)$, from the moments of $f(x)$. This method is applied to obtain the Gram-Charlier approximation to the function $G(s)$ of Chapter **IV**. For the discrete case, with points equally spaced, the author employs $H_t^{(n)}(s) = \Delta_s^t\binom{n-t}{s-t}$, n, s, and t integral, whose orthogonality property $\sum_{k=0}^{n} H_k^{(n)}(s)H_t^{(n)}(k) = 0$ if $s \neq t$, 2^n if $s = t$, makes them suited to least squares approximation.

Chapter **XI** contains introductory material on simple difference equations. First the author distinguishes between particular discrete solutions, particular continuous solutions, and general solutions. Then several pages are devoted to a list of differences of functions, to be used in solving the difference equation $\Delta u_s = f(s)$, where $f(s)$ might be a polynomial, rational fraction, exponential, trigonometric, or logarithmic function. It is shown how certain difference equations are converted into exact equations by multiplication by a suitable factor. For homogeneous linear difference equations of order higher than the first, the solution by the substitution $u_s = a^s$, depends upon the solution of an algebraic equation in a. Also, it is shown how to solve a few cases of nonhomogeneous linear difference equations with constant coefficients, by means of undetermined coefficients. The author concludes by touching on the solution of linear equations, with variable coefficients, by means of factorial series, and the vanishing of all coefficients of factorials of like degree by virtue of the given difference equation.

Appendix **A** clarifies notation and symbols. Appendix **B** lists 21 reference texts, 13 tables available in book form, 9 tables in journals, and 4 bibliographies. Appendix **C** furnishes a classified guide to formulae and methods in this book. The section on **Tables** con-

tains: T. I. Binomial coefficients $\binom{n}{k}$, $k = 0(1)10$, $n = 1(1)20$. T. II. Interpolation coefficients $\binom{s}{k}$ for Newton's binomial interpolation formula, $k = 2(1)5$, $s = [0(.01)1; 5D]$. T. III. Everett's interpolation coefficients $\binom{s+1}{3}$, $\binom{s+2}{5}$, $s = 0(.01)1$; 5D. T. IV. Lagrange's coefficients for five equally spaced points, $L_i(s)$, $i = -2(1)2$, $s = [0(.01).5; 6D]$. T. V. Legendre's polynomials $P_m(x)$ normalized to the interval $0 \leqslant x \leqslant 1$, $m = 1(1)5$, $x = 0(.01)1$; $m = 1, 2$, exactly; $m = 3, 4, 5$, 5D. T. VI. Orthogonal polynomials $P_j(s)$, (for sums, properties in Chapter IX), for $n + 1$ equally spaced points, given exactly for $j = 1(1)5$, $n = 5(1)20$, $s = 0(1)n$. T. VII. Integrals of binomial coefficients $\int_0^s \binom{t}{k} dt$, $k = 0(1)9$, $s = -1, 1(1)8$; exactly. T. VIII. Gamma function $\Gamma(x + 1)$ and digamma function $d \ln \Gamma(x + 1)/dx$ (also known as the "psi function"), for $x = [0(.02)1; 5D]$.

HERBERT E. SALZER

NBSCL

638[H, L].—R. WESTBERG, "On the harmonic and biharmonic problems of a region bounded by a circle and two parallel planes," *Acta Polytechnica*, Stockholm, no. 18, 1948, 66 p. 17.5 × 24.6 cm. Also Physics and Appl. Math. series v. 1, no. 3. Also as Ingeniörsvetenskapsakademiens *Handlingar*, no. 197.

On p. 61 are tables of $I_m = \int_0^\infty t^m dt/\Sigma$, 5S; $J_m = \int_0^\infty t^m e^{-2t} dt/\Sigma$, 1–5S; $K_m = I_m + 2I_{m+1} - J_m$, 5–6S; $m = 1(2)15$, $\Sigma = \sinh 2t + 2t$. This table is an abridgement of one given by HOWLAND [1] for $m = 1(1)20$.

On p. 64 are given 9D values of the first 11 zeros[2] z_n, $n = 0(1)10$, of $\sinh z + z$, each to 9D. For example:

$$z_0 = 2.250728611 + i\,4.212392231$$
$$z_{10} = 4.907438417 + i\,67.471628635$$

Extracts from text

[1] R. C. J. HOWLAND, "On the stresses in the neighborhood of a circular hole in a strip under tension," R. Soc. London, *Trans.*, v. 229A, 1930, p. 67; correction v. 232A, 1933, p. 169.

[2] The first four zeros were given, 4–5S, by F. SEEWALD, "Die Spannung und Formänderungen von Balken mit rechtwinkligem Querschnitt," Aerodynam. Inst. Aachen, *Abhand.*, Heft 7, 1927, p. 16. These values differ in some cases in the third figure from those obtained by the present author.

639[I].—HERBERT E. SALZER, "Coefficients for facilitating trigonometric interpolation," *Jn. Math. Phys.*, v. 27, p. 274–278, 1949. 17.4 × 25.3 cm.

The problem of expressing the trigonometric sum,

$$f(x) = C_0 + (C_1 \cos x + S_1 \sin x) + \cdots + (C_n \cos nx + S_n \sin nx),$$

so that $f(x)$ assumes given values, $f_0, f_1, \cdots, f_{2n}$, when x assumes the values, $x_0, x_1, \cdots, x_{2n}$, leads to what is commonly called Gauss's formula for trigonometric interpolation, namely,

$$f(x) = \sum_{i=0}^{2n} \Pi_{j=0}^{\prime 2n} \sin \tfrac{1}{2}(x - x_j) f_i / \Pi_{j=0}^{\prime 2n} \sin \tfrac{1}{2}(x_i - x_j).$$

The symbol Π' has its customary significance that the factor corresponding to $i = j$ is omitted from the product.

The present paper is concerned with the tabular values of the coefficients,

$$A_i^{(2n+1)} = 1/\Pi_{j=0}^{\prime 2n} \sin \tfrac{1}{2}(x_i - x_j).$$

If the values of x_i are equally spaced, then the coefficients satisfy the relationship,

$$A_i^{(2n+1)} = A_{2n-i}^{(2n+1)}.$$

The author describes his tables as follows: "Coefficients $A_i^{(2n+1)}$ are given for the 3-, 5-, 7-, 9-, and 11-point cases, all at intervals in x equal to 1, .5, .2, .1, .05, .02, and .01; also

$A_i^{(2n+1)}$ are given for functions tabulated at $2n + 1$ equally spaced points over a range of π and $\frac{1}{2}\pi$, for $2n + 1 = 3(2)11$, since those ranges, i.e., 180° and 90°, are important for many periodic functions. All the quantities $A_i^{(2n+1)}$ are given to eight significant figures."

H. T. D.

640[J].—E. H. Copsey, H. Frazer, & W. W. Sawyer, (a) "Empirical data on Hilbert's inequality," *Nature*, v. 161, 6 March 1948, p. 361. (b) "A research project," *Math. Gazette*, v. 32, May 1948, p. iii–iv.

The problem of Hilbert's inequality is discussed in detail in *MTAC*, v. 3, p. 399–400, where the results of (a) and (b) are set forth. The tables of (a) and (b) are of the largest latent root of the matrix of the nth order for which

$$a_{ij} = (i + j - 1)^{-1}$$

(a) gives this root λ_n to 9D for $n = 1(1)5, 10, 20$ while in (b) will be found λ_n to 5D for $n = 1(1)20$.

D. H. L.

641[K, L].—Zdeněk Kopal, "A table of the coefficients of the Hermite quadrature formula," *Jn. Math. Phys.*, v. 27, p. 259–261, Jan. 1949. 17.5 × 25.3 cm. Compare *MTAC*, v. 1, p. 152–153, v. 3, p. 26.

A Gauss-type quadrature formula for an approximate evaluation of definite integrals with doubly infinite limits appears to have first been established by Gourier,[1] who proved that if $f(x)$ is a function of degree not in excess of $2n - 1$,

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx = 2^{n+1} n! \pi^{\frac{1}{2}} \sum_{i=1}^{n} f(x_i)/[H_n'(x_i)]^2, \tag{1}$$

where $H_n(x)$ denotes the polynomial defined by

$$H_n(x) = e^{x^2} d^n(e^{-x^2})/dx^n, \tag{2}$$

and the x_i's in (1) are roots of the Hermite polynomials of nth order (2). Numerical values of the Christoffel numbers

$$p_i = 2^{n+1} n!/[H_n'(x_i)]^2 \tag{3}$$

for $n = 2(1)4$ were given by Berger,[2] and a more complete set of 7D values corresponding to $n = 2(1)9$ was later completed by Reiz;[3] the latter's paper was the first one in which the respective Christoffel numbers were given in decimal form.

In certain computations performed recently at the MIT, a need arose for the Christoffel numbers of the quadrature formula (1) corresponding to $n = 10(1)20$. Since their values do not appear to have been evaluated before and are apt to be frequently needed in the future, a 6D table for x_i and a 6–13D table for p_i in this range are given.

In computing the Christoffel numbers by equation (3), use has been made of values of the roots of the Hermite polynomials published previously by Smith;[4] their accuracy of 6D imposed the limit to the accuracy with which the corresponding Christoffel numbers could be evaluated. The error of no published value of p is expected to exceed one unit of the last place.

Extracts from the text

[1] G. Gourier, Acad. d. Sciences, Paris, *Comptes Rendus*, v. 97, 1883, p. 79–82.
[2] A. Berger, K. Vetenskaps Societen i Upsala, *Nova Acta*, s. 3, v. 16, no. 4, 1893, p. 3.
[3] A. Reiz, *Arkiv f. Math. Astr. och Fysik*, v. 29A, no. 29, 1943, p. 6.
[4] E. R. Smith, *Amer. Math. Mo.*, v. 43, 1936, p. 354.

642[**K, L, M**].—J. Barkley Rosser, *Theory and Application of* $\int_0^z e^{-x^2}dx$ and $\int_0^z e^{-p^2y^2}dy \int_0^y e^{-x^2}dx$. *Part I. Methods of Computation.* Brooklyn 4, N. Y., Mapleton House, 5415 Seventeenth Ave., 1948, iv, 192 p. 13.8 × 21.6 cm. $8.00. Offset print, bound in cloth.

This is a second edition of the quarto-format report, published in November 1945, which has been already reviewed in *MTAC*, v. 2, p. 213f. In smaller format the reprint is practically identical with the original, and the smaller number of pages was made possible by five times putting the material of two pages of the first edition on one in the second.

The volume contains a survey of various methods of calculating the integrals mentioned in the title. Known methods are discussed and new ones developed. The accuracy of various methods is subjected to a painstaking analysis. The author's attention is focussed on complex values of the variables, and a great number of new asymptotic developments are derived.

The book contains two tables, each covering four functions. Put

$$Rr(u) = \tfrac{1}{2}\cos \tfrac{1}{2}\pi u^2 - \tfrac{1}{2}\sin \tfrac{1}{2}\pi u^2 + \int_0^u \sin \tfrac{1}{2}\pi(u^2 - x^2)dx,$$

$$Ri(u) = \tfrac{1}{2}\cos \tfrac{1}{2}\pi u^2 + \tfrac{1}{2}\sin \tfrac{1}{2}\pi u^2 + \int_0^u \cos \tfrac{1}{2}\pi(u^2 - x^2)dx.$$

Table 1 gives values, to 12D, for $Rr(u)$, $Ri(u)$, $Rr^2(u) + Ri^2(u)$, and $\int_0^u Rr(u)dx$. The range covered is $-.06(.02) + 3(.05)5.15$, except for the integral, for which the last three entries are missing.

Table 2 covers the following four functions to 10D:

$$e^{-w^2}\int_0^w e^{y^2}dy, \quad e^{w^2}\int_w^\infty e^{-y^2}dy, \quad \int_0^w e^{-y^2}dy \int_0^y e^{x^2}dx, \quad \int_0^w e^{y^2}dy \int_y^\infty e^{-x^2}dx.$$

The first is for $w = -.2(.05)4(.1)6.5(.5)12.5$. The second is for $w = -.2(.05)3.8(.1)6.3$; the third and fourth are for $w = -.2(.05) + 3.5(.1)6$.

The book reviewed is "Part I." Portions of Part II, which are unrestricted, have been expanded and completely rewritten and now appear as chapters III and IV of *Mathematical Theory of Rocket Flight* by J. B. Rosser, Robert R. Newton, & George L. Gross, New York, McGraw-Hill, 1947, viii, 276 p.

Will Feller

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643[**L**].—J. Deschodt, *Arcs-Aires-Volumes, Centres de Gravité, Moments d'Inertie.* Paris, Office National d'Études et de Recherches Aéronautiques (ONERA), 3, rue Léon-Bonnat, 1948, Div. no. 3, vi, 73 p. 21.2 × 27 cm. Offset print from ms.

A useful summary of formulae indicated by the title. Each formula is accompanied by a figure clearly indicating the meaning of every formula element. There are more than seventy formulae for moments of inertia of plane surfaces and of solids.

644[**L**].—Harvard University, Computation Laboratory, *Annals*, v. 11: *Tables of the Bessel Functions of the First Kind of Orders Forty through Fifty-One.* Cambridge, Mass., Harvard Univ. Press, 1948, x, 620 p. The volume is dated 1948, although not published until March 1949. 19.5 × 26.7 cm. $10.00. Compare *MTAC*, v. 2, p. 176f, 261f, 344; v. 3, p. 102, 117–118, 185–186, 367.

This is the ninth of the thirteen planned volumes of the monumental edition of tables of Bessel functions, up to and including $J_{100}(x)$, prepared at the Harvard Computation Laboratory. Practically all of the results set forth in this volume are entirely new. We are given $J_{40}(x) - J_{51}(x)$ for $x = [18.38(.01)99.99; 10D]$; the first significant values, .00000 00001, for $J_{51}(x)$ are when $x = 26.75(.01)27.40$ inclusive.

In K. Hayashi, *Tafeln der Besselschen, Theta-, Kugel-, und anderer Funktionen*. Berlin, 1930, are the following values of $J_n(x)$: $n = 40$–51, $x = 20$, and at least 23D; $x = 30$, and at least 27D; $x = 40$, and at least 32D; $x = 50$, and at least 29D.

The values of Hayashi, rounded to 10D, in every case agree with those in the Harvard volume.

John A. Harr designed the sequence control tapes, supervised the computation, and prepared the manuscript for publication.

R. C. A.

645[L].—G. G. Macfarlane, "The application of Mellin transforms to the summation of slowly convergent series," *Phil. Mag.*, s. 7, v. 40, Feb. 1949, p. 188–197. 17.2 × 25.2 cm.

Many problems in applied mathematics are reduced to the numerical calculation of a series. To those who have spent any time with such problems, it is common knowledge that the series oftentimes converge slowly in some sense or other. That is, the series might be of the nature

$$(1) \qquad \sum_{n=1}^{\infty} 1/n^{\alpha}, \quad 1 < \alpha < 2$$

and the time necessary to sum such a series to a given number of significant figures can be prohibitive. On the other hand for such a simple series as (1) (which is, incidently $\zeta(\alpha)$, where ζ is the Riemann zeta function), the method of Euler serves quite adequately. The author discusses here a method which is particularly useful for the summing of series of the form

$$(2) \qquad \sum_{n=0}^{\infty} f[(n+a)^{r}]$$

presuming, of course, that $f(x)$ is sufficiently well behaved to accept the Mellin transform and that the interchange of summation and integration at certain stages of his procedure is permissible.

Subject to appropriate conditions, the Mellin transform pair is

$$(1) \qquad F(s) = \int_0^{\infty} f(x)\cdot x^{s-1}dx$$

and

$$(2) \qquad f(x) = (2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)x^{-s}ds, \quad \sigma_1 < \sigma < \sigma_2$$

where $\sigma = Re(S)$ and the σ_1 and σ_2 define the abscissae of convergence of the integral (1). [See E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937]. From (2) we note that for n integer

$$f(n+a) = (2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)(n+a)^{-s}ds$$

and hence

$$(3) \qquad \sum_{n=0}^{\infty} f(n+a) = (2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\zeta(s,a)ds, \quad \sigma_1 < \sigma < \sigma_2,$$

where

$$\zeta(s,a) = \sum_{n=0}^{\infty} (a+n)^{-s}, \quad Re(s) > 1$$

is the generalized Riemann zeta function. It is, of course, assumed that interchange of sum and integral is permissible. By various devices known to those versed in the methods of contour integration and by use of the properties of $\zeta(s,a)$ it is possible to do something about the evaluation of the integral in equation (3) for a known function $f(x)$.

The author considers three examples of this procedure. First, the well-known sum

$$\sum_{n=1}^{\infty} (\cos ny)/n^2$$

is treated. Here $f(x) = \cos x$ and hence

$$F(s) = 2^{s-1}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}s)/\Gamma(\tfrac{1}{2} - \tfrac{1}{2}s). \quad 0 < \sigma < 1$$

Therefore $\cos x/x^2$ has the transform

$$F(s) = 2^{s-3}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}s - 1)/\Gamma(\tfrac{3}{2} - \tfrac{1}{2}s). \quad 2 < \sigma < 3$$

Hence

$$\sum_{n=1}^{\infty} (\cos ny)/(ny)^2 = (2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty} 2^{s-2}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}s - 1)y^{-s}\zeta(s)ds/\Gamma(\tfrac{3}{2} - \tfrac{1}{2}s), \quad 2 < \sigma < 3.$$

Since the integrand has simple poles at $s = 0$, 1 and 2 and behaves properly at infinity, we get

$$\sum_{n=1}^{\infty} (\cos ny)/(ny)^2 = \pi^2/(6y^2) - \pi/(2y) + \tfrac{1}{4}.$$

A second series is

$$\sum_{n=0}^{\infty} (-1)^n(2n+1)^{-1} \quad J_1[(2n+1)y], \tag{4}$$

where $J_1(y)$ is the finite Bessel function of the first order. Here we get for $y < 1$, the rapidly convergent series

$$-\sum_{n=0}^{\infty} \frac{(-)^n}{2} \frac{B_{2n+1}(\frac{1}{4})(2y)^{2n+1}}{(2n+1)\Gamma(n+1)\Gamma(n+2)}.$$

The $B_n(a)$ are the Bernoulli polynomials of order n.

A final example is

$$\sum_{m=1}^{M} (1 - xm^{\frac{1}{2}})^{\frac{1}{2}}/m^{\frac{1}{2}},$$

where M is the largest integer such that $xM^{\frac{1}{2}} < 1$. In this case we get an asymptotic series for small x, that is M large. The answer in this case is expressed in terms of $\zeta(s, 1)$ and powers of x. The technique employed in the second and third examples is similar to the one employed in the first one. Observe that the summing of the series (4) involves terms which are

$$O[(2m+1)^{-\frac{3}{2}}(-)^m \cos\{(2m+1)y - 3\pi/4\}]$$

for m sufficiently large and hence great difficulties would be encountered in getting the sum of the series directly from (4).

The note ends with a table of 72 Mellin transforms which, of course, is equivalent to a table of bilateral Laplace transforms under proper substitution.

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Editorial Note: The origin of the transforms of Robert Hjalmar Mellin (1854–1933) is indicated on p. 7 of the work of Titchmarsh referred to above. The idea of the reciprocity exhibited in (1)–(2) above, occurs in the famous memoir on prime numbers by G. F. Bernhard Riemann (1826–1866), "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse," Preuss. Akad. d. Wissen. zu Berlin, *Monatsberichte*, 1859, p. 671–680; Riemann, *Werke*, Leipzig, 1876, p. 136–144. It was formulated explicitly by Eugène Cahen (1865–) in his doctoral diss., "Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues," *Ann. de l'École Norm. Sup.*, s. 3, v. 11, 1894, p. 75–164. But the first accurate discussion was given by Mellin: (i) "Ueber die fundamentale Wichtigkeit des Satzes von Cauchy für die Theorien der Gamma- und der hypergeometrischen Funktionen," Finska

Vetenskaps Societeten, *Acta*, v. 21, no. 1, 1896, p. 1–115; (ii) "Über den Zusammenhang zwischen den linearen Differential- und Differenzengleichungen," *Acta Math.*, v. 25, 1902, p. 139–164.

646[L].—K. MITCHELL, "Tables of the function $\int_0^x - \log|1 - y|dy/y$, with an account of some properties of this and related functions," *Phil. Mag.*, s. 7, v. 40, Mar. 1949, p. 351–368. 17.2 × 25.6 cm. Compare *MTAC*, v. 1, p. 189, 457–459; v. 2, p. 180, 278.

Let $f(x) = \int_0^x - \ln|1 - y|dy/y = - Rl(1 - x) = \zeta(1, 2|x)$, $Rl(x) = \int_1^x \ln t dt/(t - 1) = - f(1 - x)$. Some properties of an integral of this type have been given by POWELL [1] with a table of the function for $x = [0(.01)2(.02)6; 7D]$. An earlier table, NEWMAN,[2] gives the function $Rl(1 \pm x)$, for $x = [0(.01).5; 12D]$, and a comparison of these two tables over their common range (FLETCHER [3]) reveals discrepancies which are traced to errors in Newman's table. Powell's table, over the common range, contains only rounding errors in the last decimal place. Another table (SPENCE,[4] 1809) gives $Rl(x)$, $x = [1(1)100; 9D]$; and yet another (KUMMER,[5] 1840), gives a function akin to $f(x)$ for $x = [- 11(1) + 10; 11D]$. These tables offer only isolated points of comparison.

The author's attention was drawn to the function by a physical problem in 1940, and the tables of $f(x)$, for $x = [- 1(.01) + 1; 9D]$, $x = [0(.001).5; 9D]$ were calculated and are given in this paper, p. 357–363. Comparison of the author's table with that of Powell reveals one error in the latter; at $x = .01$ (Powell's variable), *for* 1.5886 274, *read* 1.5886 254. There are several other rounding errors in the last decimal place, which have not been separately listed. Errors in Newman's table are listed by Fletcher.

Extracts from text

[1] E. O. POWELL, *Phil. Mag.*, s. 7, v. 34, 1943, p. 600–607.
[2] F. W. NEWMAN, *The Higher Trigonometry. Superrationals of Second Order.* Cambridge, 1892.
[3] A. FLETCHER, *Phil. Mag.*, s. 7, v. 35, 1944, p. 16–17.
[4] W. SPENCE, *An essay on the Theory of the Various Orders of Logarithmic Transcendents*, London, 1809, p. 24.
[5] E. E. KUMMER, *Jn. f. d. r. u. angew. Math.*, v. 21, 1840, p. 74–90, 193–225, 328–371.
EDITORIAL NOTE. The table in question is on p. 88 and the function is $\Lambda(x) = \int_0^x \ln(1+t)dt/(1 + t)$. $\Lambda(x) = 0$, for $x = 4.50374185563$.

647[L].—NBSCL, *Tables of Bessel Functions of Fractional Order. Volume II.* New York, Columbia University Press, 1949, xviii, 365 p. Offset print. 20 × 26.6 cm. $10.00. The foreword by Prof. R. E. LANGER occupies p. vii–x. Introduction, p. xiii–xvii, by MILTON ABRAMOWITZ. Compare *MTAC*, v. 1, p. 93, 300; v. 3, p. 187, 339.

The present volume, devoted to the tabulation of $I_\nu(x)$ for $\pm \nu = \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$, is a sequel to the volume of 1948 containing $J_n(x)$ for the same orders. The functional values in both volumes are given either to 10D or 10S. The interval in the argument has been so chosen that interpolation with the aid of the tabulated second central differences will yield the maximum attainable accuracy over most of the range covered. In some regions, where this desideratum is not met, fourth central differences are also given; in such regions it is always possible to obtain an accuracy of at least 7S by Everett's interpolation formula involving only second differences. Interpolation is not feasible close to the origin (for $x < .05$), and therefore in this region the auxiliary function $x^{-\nu}I_\nu(x)$ has been tabulated, together with its second central differences.

The tables of $I_\nu(x)$ cover a range of x from 0 to 25. The function $e^{-x}I_\nu(x)$ is tabulated for $x = 25(.1)50(1)500(10)5000(100)10000(200)30000$. With the aid of these values it is possible to compute $I_\nu(x)$ in this range of x. For $x > 30000$ an accuracy of at least 9S can be obtained with the first two terms of the asymptotic expressions for $I_\nu(x)$.

Values of $I_{-\nu}(x)$ are tabulated up to $x = 13$ only, since for larger values of x they are identical with those of $I_\nu(x)$ to 10S. For $\nu = -\frac{3}{4}, -\frac{2}{3}$, in tables of $I_\nu(x)$, $x = 0(.001)1(.01)13$;

for $\nu = -\frac{1}{3}, -\frac{1}{4}$, $x = 0(.001).8(.01)13$; for $\nu = \frac{1}{4}, \frac{1}{3}$, $x = 0(.001).6(.01)25$; for $\nu = \frac{2}{3}, \frac{3}{4}$, $x = 0(.001).5(.01)25$.

On p. xviii are 15D values of constants including 16 involving the gamma function. Tables of Everett Interpolation Coefficients are given p. 333–343, and of $L_\nu(\mu)$ for interpolation in the ν direction, p. 345–365.

Extracts from text

EDITORIAL NOTE: Except for small unreliable tables by DINNIK these valuable tables are the first published tables of their kind.

648[L].—NBSINA, *Tables of* $I_0(2\sqrt{x})$, $I_1(2\sqrt{x})/\sqrt{x}$, $K_0(2\sqrt{x})$, $K_1(2\sqrt{x})/\sqrt{x}$ *and Related Functions.* $0 \leqslant x \leqslant 410$. Computed under the direction of Dr. GERTRUDE BLANCH, Institute for Numerical Analysis, U.C.L.A., Los Angeles, California, February 1949. v, 26 leaves, with text on only one side. Our leaf numbers do not agree with those of the text because there are no leaves ii or 1 in the original. 20.17 × 35.7 cm. These tables are a sequel to the tables reviewed in RMT **505**, v. 3, p. 107.

T. I is of $I_0(2x^{\frac{1}{2}})$, $I_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}}$, for $x = [0(.02)1.5(.05)6.2$; 8S or 9S$]$, δ^2, second central differences. T. II. $e^{-2x^{\frac{1}{2}}}I_0(2x^{\frac{1}{2}})$, $e^{-2x^{\frac{1}{2}}}I_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}}$, for $x = [6.2(.1)13(.2)36(.5)115(1)160(5)410$; 7S or 8S$]$, δ^2 or modified δ^2. T. III. $K_0(2x^{\frac{1}{2}})$, $K_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}}$, for the same ranges as T. I, and 7S or 8S, δ^2 or modified δ^2, except for a small region near the origin. T. IV. $e^{2x^{\frac{1}{2}}}K_0(2x^{\frac{1}{2}})$, $e^{2x^{\frac{1}{2}}}K_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}}$, for the same ranges as T. II, and 7S or 8S, δ^2 or modified δ^2.

For large values of the argument, $I_0(y)$ and $I_1(y)$ are of the order of magnitude of e^y, while $K_0(y)$ and $K_1(y)$ behave like e^{-y}. For $x \geqslant 6.2$ (T. II and III) the tabulated functions have been multiplied by appropriate exponential factors in order to retain the interpolable character of the table.

Let $u = F_n(x)$ or $G_n(x)$, where $F_n(x) = I_n(2x^{\frac{1}{2}})/x^{\frac{1}{2}n}$; $G_n(x) = K_n(2x^{\frac{1}{2}})/x^{\frac{1}{2}n}$, and $I_n(t)$ and $K_n(t)$ are Bessel Functions of order n. The u satisfies the differential equation

$$x\, d^2u/dx^2 + (n+1)du/dx - u = 0.$$

If, in $F_n(x)$ and $G_n(x)$, the functions $I_n(2x^{\frac{1}{2}})$ and $K_n(2x^{\frac{1}{2}})$ are replaced by $J_n(2x^{\frac{1}{2}})$ and $Y_n(2x^{\frac{1}{2}})$, respectively, we obtain $w_n(x)$ which satisfies the Bessel-Clifford differential equation (see RMT **505**)

$$x\, d^2w/dx^2 + (n+1)dw/dx + w = 0.$$

The functions $K_0(2x^{\frac{1}{2}})$ and $K_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}}$ have singularities at $x = 0$, and interpolation in the region close to the origin, where differences are not given, can be performed more advantageously in at least two other available tables, namely:

(a) *Tables of the Bessel Functions* $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, 1948, for $0 \leq x \leq 1$, 7S in $K_0(x)$, $K_1(x)$; *MTAC*, v. 3, p. 187.

(b) BAASMTC, *Mathematical Tables*, v. VI, Part I, 1937; auxiliary functions E_0, F_0, E_1, F_1, $x = [0(.01).5$; 8D$]$; *MTAC*, v. 1, p. 361–363.

For $x < 100$ the entries were computed by interpolation from (b). A few entries near $x = 0$ in T. III were computed otherwise.

In the following regions there may be an error of two units in the last place [the pages are those as corrected]:

Table I. $I_0(2x^{\frac{1}{2}})$, entries on p. 1, 3–11; $x^{-\frac{1}{2}}I_1(2x^{\frac{1}{2}})$, p. 11.

Table III. Entries on p. 14, both functions.

Table IV. All entries of $e^{2x^{\frac{1}{2}}}K_0(2x^{\frac{1}{2}})$, and entries of

$$e^{2x^{\frac{1}{2}}}K_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}} \text{ on p. 22–24, } x < 100.$$

All other entries should be correct to within a unit of the last place, and for $x > 100$ to within .6 units.

Extracts from text

649[L].—Lidia Stankiewicz, "Sul calcolo della piastra poggiata su suolo elastico," Accad. Naz. d. Lincei, *Atti, Rendiconti, cl. d. sci. fis. matem. e nat.*, s. 8, v. 5, Dec. 1948, p. 339–344. 18 × 26.6 cm.

$F(x, y) = \int_0^\infty \int_0^\infty \sin (xu) \sin (yv) du\, dv/[uv\{(u^2 + v^2)^2 + 1\}]$ is here tabulated for x and $y = [0(.5)8(1)12, \infty; 3D]$.

650[L, M].—David L. Arenberg & Doris Levin, *Table of Fresnel Integrals and Derived Functions.* Naval Research Laboratory Field Station, 470 Atlantic Avenue, Boston, Mass., [1948], [15] leaves, hektographed. 20.5 × 26.7 cm. Not available for general distribution.

In this publication are given 4D values of

$$C(u) = \int_0^u \cos (\tfrac{1}{2}\pi x^2)dx, \quad S(u) = \int_0^u \sin (\tfrac{1}{2}\pi x^2)dx,$$

for $u = 0(.1)20$, and for $u = 8(.02)16$. The values of $C(u)$ and $S(u)$ for $u \leq 8$ were taken from the tables of Sparrow.[1] For $u > 8$ computations were made by means of approximate formulae from the semi-convergent series given by Watson:[2]

$$(1) \qquad C(u) \approx \tfrac{1}{2} + (2\pi z)^{-\frac{1}{2}}[\sin z - (2z)^{-1} \cos z],$$

$$(2) \qquad S(u) \approx \tfrac{1}{2} - (2\pi z)^{-\frac{1}{2}}[\cos z + (2z)^{-1} \sin z],$$

where $z = \frac{1}{2}\pi u^2$. For $z > 50$, (1) and (2) are accurate to ± 1 in the fourth decimal place. All other functions of $C(u)$ and $S(u)$ will have corresponding errors.

For $u = 0(.1)20$, there are 4D tables of $[C(u)]^2$, $[S(u)]^2$, $R_{00}(u) = \{[C(u)]^2 + [S(u)]^2\}^{\frac{1}{2}}$, and $R_{\frac{1}{2}\frac{1}{2}}(u) = \{[C(u) - \frac{1}{2}]^2 + [S(u) - \frac{1}{2}]^2\}^{\frac{1}{2}}$; and 5D or 6D tables of $[C(u) - \frac{1}{2}]^2$ and $[S(u) - \frac{1}{2}]^2$. The function $R_{\frac{1}{2}\frac{1}{2}}(u)$ was computed mainly to obtain an independent check of the consistency of the calculations, since this is a monotonic function giving the relative distance between points on the Euler spiral (often called after Cornu) and the limiting value $C(\infty) = S(\infty) = \pm \frac{1}{2}$. By taking differences between the tabulated values of $R_{\frac{1}{2}\frac{1}{2}}(u)$, one easily detected flagrant discrepancies in $S(u)$ and $C(u)$.

These tables were computed to meet the need of more extended tables than those already available.

Extracts from introductory text

[1] C. M. Sparrow, *Table of Fresnel Integrals*, Rouss Physical Laboratory, University of Virginia, 1934.

Editorial Note: This publication, photo-lithoprint reproduction of author's manuscript by Edwards Bros., Ann Arbor, Mich., contains ii, 9 p. (21 × 27.1 cm.), and the tables are of $C(u)$ and $S(u)$, for $u = [0(.005)8; 4D]$. The first paragraph of the text by the author, Carroll Mason Sparrow, 1880–1946, is as follows: "The greater part of the following table was made some years ago in connection with a study of imperfect gratings; existing tables not being adequate by reason of their too large interval. This manuscript table was resurrected and slightly extended to meet a teaching need, and is here reproduced in the hope that it may prove of some use to others." The Arenberg & Levin tables of $C(u)$ and $S(u)$, for $u \leq 8.5$ are identical with those given in Jahnke & Emde, *Tables of Functions*, 1945, p. 34. In no previously published tables has u been greater than 8.5.

[2] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, second ed., Cambridge and New York, 1944, p. 545.

651[L, M].—Harvard University, Computation Laboratory, *Annals*, v. 18, 19: *Tables of Generalized Sine- and Cosine-Integral Functions*, Parts I and II. Cambridge, Mass., Harvard Univ. Press, 1949, xxxviii, 462, viii, 560 p. 20 × 26.7 cm. $10.00 + $10.00.

The Computation Laboratory has here undertaken to present as complete and useful a table of certain integrals as can be included within the scope of 1000 pages. The inte-

grals are

$$
(1) \quad \begin{cases} S(a, x) = \int_0^x \sin u \, dx/u, \quad C(a, x) = \int_0^x (1 - \cos u) dx/u, \\ \bar{C}(a, x) = \int_0^x \cos u \, dx/u = \sinh^{-1}(x/a) - C(a, x), \\ Ss(a, x) = \int_0^x \sin u \sin x \, dx/u, \quad Sc(a, x) = \int_0^x \sin u \cos x \, dx/u, \\ Cs(a, x) = \int_0^x \cos u \sin x \, dx/u, \quad Cc(a, x) = \int_0^x \cos u (1 - \cos x) dx/u, \\ \bar{C}c(a, x) = \int_0^x \cos u \cos x \, dx/u = \sinh^{-1}(x/a) - C(a, x) - Cc(a, x), \end{cases}
$$

where $u = (x^2 + a^2)^{\frac{1}{2}}$. Clearly, tables of these functions need include only one of the integrals C, $\bar{C}$, and only one of the integrals Cc, $\bar{C}c$; C and Cc were chosen because they afford much better interpolation near the origin.

The tabulation extends over a set of points (a, x) within the square $0 \leq a \leq 25$, $0 \leq x \leq 25$. The set was so chosen as to provide for interpolation in 6D tables within as large as possible a portion of the square, and yet permit the tables to be encompassed in two volumes.

These volumes are arranged in sections, each section consisting of the complete table, ordered by ascending x, for one value of the parameter a. The sections (about 350) are in turn ordered by ascending a, and those sections corresponding to $0 \leq a < 2$ are contained in Part I, while those corresponding to $2 \leq a \leq 25$ are in Part II.

When $a = 0$, the integrals $S(a, x)$, $C(a, x)$ reduce to the sine-integral and cosine-integral functions

$$
Si(x) = \int_0^x \sin x \, dx/x, \quad Ci(x) = \int_0^x \cos x \, dx/x.
$$

These functions are classical, thorough studies having been made [1] by 1906. Further they have been exhaustively tabulated.[2] On the other hand, for non-zero a the previously existing tabulations are fragmentary and inadequate.[3–6] The present volumes were prepared with the thought in mind that the class of tabulated functions should be augmented to include the composite functions, that 6D accuracy should be provided, that the domain of tabulation should be large enough to cover the cases that arise in practice, and that the mesh should be fine enough to admit interpolation to the accuracy that the applications demand.

When $a = 0$,

$$
\begin{aligned}
S(0, x) &= Si(x), \quad C(0, x) = \gamma + \ln x - Ci(x), \\
Sc(0, x) &= \tfrac{1}{2}S(0, 2x) = \tfrac{1}{2}Si(2x), \quad Cs(0, x) = \tfrac{1}{2}S(0, 2x) = \tfrac{1}{2}Si(2x), \\
Ss(0, x) &= \tfrac{1}{2}[\ln 2 + \gamma + \ln x - Ci(2x)] = \tfrac{1}{2}C(0, 2x), \\
Cc(0, x) &= \tfrac{1}{2}[\ln 2 - \gamma - \ln x - Ci(2x)] + Ci(x) = \tfrac{1}{2}C(0, 2x) - C(0, x),
\end{aligned}
$$

where γ is Euler's constant.

When $a \neq 0$, it may be shown that

$$
\begin{aligned}
Sc(a, x) &= \tfrac{1}{2}[Si(z) - Si(y)], \quad Cs(a, x) = \tfrac{1}{2}[Si(z) + Si(y)] - Si(a), \\
Ss(a, x) &= -\tfrac{1}{2}[Ci(z) + Ci(y)] + Ci(a), \quad \bar{C}c(a, x) = \tfrac{1}{2}[Ci(z) - Ci(y)],
\end{aligned}
$$

where $z = u + x$, $y = u - x$.

$$
\int_0^\infty \sin u \, dx/u = \tfrac{1}{2}\pi J_0(a), \quad \int_0^\infty \cos u \, dx/u = \tfrac{1}{2}\pi Y_0(a).
$$

The six functions (1) [omitting $\bar{C}$ and $\bar{C}c$] can be represented as a combination of elementary functions and power series in a and x, convergent for all values of a and x. For these power series in x, 10D tables of the first four or five of the various coefficients $\alpha_j(a)$, $\beta_j(a)$, $\gamma_j(a)$, $\nu_j(a)$, $\lambda_j(a)$, $\mu_j(a)$ are given (p. xxxvi–xxxviii), for $a = 0(.01).99$.

In the half-unit rectangle $0 \leq a \leq 1$, $0 \leq x \leq \frac{1}{2}$, the integrals were computed by means of these series. The error in the coefficients is in all cases less than 6×10^{-9}; the resulting

error in the integrals is of lower order than the truncation error. The introduction was written by Mr. SINGER with the help of Mr. GADD. "Computation of the tables," p. xix–xxv.

"Interpolation" by J. ORTEN GADD, JR. & THEODORE SINGER (p. xxvi–xxx). Throughout the tables the remainder after linear interpolation is less than 5 units in the third decimal place, and the remainder after second order interpolation is less than 1.2 units in the third decimal place. This is true regardless of whether the interpolation is accomplished by means of a two-way formula or by several uni-variate interpolations. Actually, the remainders after interpolation are much smaller throughout the greater part of the tables.

Of the eight integrals (1) six are tabulated in these volumes. The others $\bar{C}(a, x)$ and $\bar{C}c(a, x)$ are readily determined from $C(a, x)$, $Cc(a, x)$, and $\sinh^{-1}(x/a)$. It is anticipated that tables of the inverse hyperbolic functions will appear in the *Annals*, v. 20.

"Applications" by RONOLD W. P. KING (p. xxxi–xxxv).

The following list is partly representative of the types of problems that have been or may be investigated by using one-dimensional Helmholtz integrals which lead to the generalized sine- and cosine-integral functions.

(a). Vector and scalar potentials of electric circuits; open-wire transmission lines; linear, loop, and rhombic antennae, and arrays using these as elements. Distributions of current may be uniform, sinusoidal, or exponential.
(b). Self-impedance (including radiation resistance) of a polygonal loop antenna that is not so small that retardation is negligible.[7]
(c). Mutual impedance of rectangular loop antennae separated by an arbitrary distance.[7]
(d). The general analysis of two-wire, four-wire, multi-wire, and polyphase transmission lines, including the determination of radiation resistance.[7,8]
(e). Radiation resistance of a linear radiator with a sinusoidal distribution of current.[9]
(f). Distribution of current and impedance for a cylindrical antenna.[10–17]

Extracts from introductory text

[1] NIELS NIELSEN, *Theorie der Integrallogarithmus*. Leipzig, 1906; the author gives an exhaustive bibliography.

[2] NBSMTC, *Tables of Sine, Cosine, and Exponential Integrals*. v. 1–2, New York, 1940; *Table of Sine and Cosine Integrals for Arguments from 10 to 100*. New York, 1942. The references contained in these volumes include a full bibliography of tables of these functions.

[3] C. J. BOUWKAMP, "Note on an integral occurring in antenna theory," Natuurkundig Laboratorium de N. V. Philips' Gloeilampenfabrieken, Eindhoven, Netherlands, Unpubl. ms.

[4] R. V. D. CAMPBELL, "Evaluation of the function $S(b, h) = \int_0^h \sin (x^2 + b^2)^{\frac{1}{2}}dx/(x^2 + b^2)^{\frac{1}{2}}$" June, 1944; [see *MTAC*, v. 2, p. 218].

[5] H. A. ARNOLD, R. V. D. CAMPBELL, & R. R. SEEBER, JR., "Evaluation of the function $C(b, h) = \int_0^h \cos (x^2 + b^2)^{\frac{1}{2}}dx/(x^2 + b^2)^{\frac{1}{2}}$," Oct., 1944; [see *MTAC*, v. 2, p. 218].

[6] Curves of some of these functions appear in an article by CHARLES W. HARRISON, JR., "A note on the mutual impedance of antennas," *Jn. Appl. Physics*, v. 14, June 1943, p. 306–309.

[7] R. W. P. KING, *Electromagnetic Engineering*, New York, v. 1, 1945, p. 408, 426, 478f.

[8] C. T. TAI, "Theory of coupled antennas and its application," Diss. Harvard, 1947.

[9] JULIUS A. STRATTON, *Electromagnetic Theory*. New York, 1941, p. 444, and R. W. P. KING[7], p. 565.

[10] M. ABRAHAM, "Die elektrischen Schwingungen um einen stabförmigen Leiter, behandelt nach der Maxwell'schen Theorie," *Annalen d. Physik*, v. 302 or n.s., v. 66, 1898, p. 435–472.

[11] C. W. OSEEN, "Über die electromagnetischen Schwingungen an dünnen Stäben," *Arkiv f. Mat. Astron. o. Fysik*, v. 9, no. 30, 1914, 27 p.

[12] ERIK HALLÉN, (i) "Theoretical investigations into the transmitting and receiving qualities of antennae," K. Vetenskaps Soc. i Upsala, *Nova Acta*, s. 4, v. 11, no. 4, 1938, 44 p.; (ii) "Iterated sine and cosine integrals," R. Inst. Techn., Stockholm, *Trans.*, v. 9, no. 12, 1947; (iii) "On antenna impedances," *Trans.*, no. 13, 1947.

[13] L. V. KING, "Radiation field of a perfectly conducting base insulated cylindrical aerial over a perfectly conducting plane earth and the calculation of radiation resistance and reactance," R. Soc. London, *Trans.*, v. 236A, 1937, p. 381–422.

[14] R. W. P. KING & F. G. BLAKE, JR., "The self-impedance of a straight symmetrical antenna," Inst. Radio Engin., *Proc.*, v. 30, 1942, p. 335–349.

[15] R. W. P. King & Charles W. Harrison, Jr., (i) "The distribution of current along a symmetrical center-driven antenna," I.R.E., *Proc.*, v. 31, 1943, p. 548–567; (ii) "The impedance of short, long, and capacitively loaded antennas with a critical discussion of the antenna problem," *Jn. Appl. Physics*, v. 15, 1944, p. 170–185.

[16] Marion C. Gray, "A modification of Hallén's solution of the antenna problem," *Jn. Appl. Physics*, v. 15, 1944, p. 61–65.

[17] R. W. P. King & David Middleton, (i) "The cylindrical antenna; current and impedance," *Quart. Appl. Math.*, v. 3, 1946, p. 302–335; (ii) "The thin cylindrical antenna: a comparison of theories," *Jn. Appl. Physics*, v. 17, 1946, p. 273–284.

652[**L, M**].—S. A. Schelkunoff, *Applied Mathematics for Engineers and Scientists.* (*The Bell Telephone Laboratories Series.*) New York, Van Nostrand, 1948, p. 456.

On this page there is a table of

$$SC(x) = \int_0^x Si\, u\, dCi u = \int_0^x Si\, u \cos u\, du/u,\ x = k\pi,$$

for $k = [.2(.2)2;\ 5D]$.

653[**L, S**].—G. H. Godfrey, "Diffraction of light from sources of finite dimensions," *Australian Jn. of Sci. Res.*, s. A. *Phys. Sciences*, Melbourne, v. 1, no. 1, Mar. 1948, p. 1–17.

T. 1, p. 7: Rectangular aperture, $I_x = [Si(2x) - (\sin^2 x)/x]/\pi$, $x = [0(.1)15(.5)34.5;\ 5D]$. T. 2, p. 8, $I_{x+2.4\pi} - I_x$, $x = [-1.2\pi(.1\pi) + 1.3\pi;\ 5D]$. T. 3, p. 9, $I_{x+3.4\pi} - I_x$, $x = [-1.7\pi(.1\pi)0;\ 5D]$. T. 4, p. 13, Circular aperture, $I_t = \int_0^t H_1(2x)dx/(\pi x^2)$, $t = [0(.1)15;\ 4D]$. T. 5, p. 15, $I_{t+3.4} - I_t$, $t = [-4.2(.1) - 2.8;\ 4D]$.

654[**M**].—Wolfgang Gröbner & Nikolaus Hofreiter, *Integraltafel. Erster Teil: Unbestimmte Integrale.* Vienna and Innsbruck, Springer-Verlag, 1949, viii, 166 p. 20.7 × 29.8 cm. $4.20. Offset print of a manuscript original. This work was first published at Braunschweig, in 1944. It is stated in the preface (p. iv) that of the first edition a French translation was made by Engineer Weber (Ministère de l'Armement S.F.I.S., Rapport Nr. 451–01–01/02/03) in which the authors of the original work are not mentioned.

This work by professors at the Universities of Innsbruck and Vienna respectively is divided into three sections, namely: 1. *Rational Integrands* (p. 1–21); 2. *Algebraic Irrational Integrands* (p. 22–106); 3. *Transcendental Integrands* (p. 107–166). Thus elliptic and hyperelliptic integrals are listed in section 2, but the Weierstrass and Jacobi elliptic functions in section 3, along with Bernoulli and Euler numbers, sine integral, cosine integral, exponential integral, etc. There are often 13–17 integrals on a page, all most neatly written, with numerous cross references, some explanations and references to discoverers, and frequent mention of Jahnke & Emde's *Funktionentafeln.* The whole presentation is exceedingly clear and satisfactory. It is remarked in the preface that the second part of the work, on Definite Integrals, is, with the first part, to form a unit, so that gaps now apparent in the first part will later be satisfactorily filled.

The authors state that the following works were at their disposal: H. B. Dwight, *Tables of Integrals and other Mathematical Data.* New York, 1934; M. Hirsch, *Integraltafeln oder Sammlung von Integralformeln.* Berlin, 1810; W. Láska, *Sammlung von Formeln der reinen und angewandten Mathematik.* Braunschweig, 1888–1894; F. Minding, *Sammlung von Integraltafeln.* Berlin, 1849; C. Naske, *Integralformeln für Ingenieure und Studierende.* Berlin, 1935; B. O. Peirce, *A Short Table of Integrals.* Third ed., Boston, 1929 ("Vorzügliches Buch"); G. Petit-Bois, *Tafeln unbestimmter Integrale.* Leipzig, 1906.

R. C. A.

655[M, P].—E. T. GOODWIN & J. STATON, "Table of $\int_0^\infty e^{-u^2}du/(u+x)$," *Quart. Jn. Mech. Appl. Math.*, Oxford, v. 1, Sept. 1948, p. 319–326.

The function $f(x) = \int_0^\infty e^{-u^2}du/(u+x)$ is tabulated for $x = [0(.02)2(.05)3(.1)10; 4D]$. To facilitate interpolation for small x an auxiliary function $g(x) = f(x) + \log x$ is tabulated for $x = [0(.1)1; 4D]$. An asymptotic expression is given for values of x greater than 10. Full details of the method of computation are set forth and an interesting application of the Euler transformation to the summation of asymptotic series is included.

The function $f(x)$ arose in research connected with the determination of the response of a detector to a random noise voltage having a narrow spectrum.

Extracts from text

656[P].—A. C. STEVENSON, "The centre of flexure of a hollow shaft," London Math. Soc., *Proc.*, s. 2, v. 50, p. 536–549, 1949.

The author re-solves the problem of flexure by a transverse force of a bar of circular cross-section with a cylindrical cavity of circular cross-section having any position and size with respect to the outer circle. The load force is assumed to act through the centroid perpendicular to the axis of symmetry of the cross-section. The general solution is obtained in bipolar coordinates in terms of the author's modification of the classical Saint-Venant flexure theory. This involves three plane harmonic functions which are written as appropriate infinite trigonometric series, of which all coefficients are determined so as to satisfy the given boundary conditions. The most complete previous solution was given by SETH;[1] this however did not consider explicitly the centre of flexure or the limiting case when the cavity just reaches the edge of the cylinder. The present paper gives, p. 548–549, 5D tables for determining the torsional moment, the associated twist, the centre of flexure, the "centre of least strain energy," and the centroid, all as functions of two dimensionless parameters which define the ratios between the radii of the inner and outer circles and the distance between their centres, $\lambda = .1(.1).9$, $\mu = .1(.1).9$, $\lambda + \mu \not> 1$. The formulae for the case when the cavity just touches the outer surface are given in terms of trigamma and tetragamma functions.[2]

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[1] B. R. SETH, "On the flexure of a hollow shaft, I–II," Indian Acad. Sci., Proc., v. 4, 1936, p. 531–541; v. 5, 1937, p. 23–31.

[2] H. T. DAVIS, *Tables of the Higher Math. Functions*, v. 2, Bloomington, Ind., 1935; BAASMTC, *Mathematical Tables*, v. 1, second ed. 1946. See *MTAC*, v. 3, p. 424.

657[S].—NBSCL, *Tables of Scattering Functions for Spherical Particles*. (NBS *Applied Mathematics Series*, no. 4, issued 25 Jan. 1949). Washington, D. C., Superintendent of Documents, 1948, xiv, 119 p. 18 × 26 cm. $0.45.

This is a collection of four sets of tables useful in the study of the scattering of electromagnetic radiation by transparent as well as absorbing and dispersing spherical particles over a wide range of ratios of particle radius to radiation wave length.

The scattering of light by particles has been of great scientific interest from the time of the early work of TYNDALL and RAYLEIGH (1870) on the blue color of the sky and on the colors produced in illuminated suspensions. The subject has attracted renewed attention in recent years from the effect of fog and rain on the action of microwave radar, as well as that of colloidal suspensions on visible light.

The tables presented here are based on the fundamental work of GUSTAV MIE,[1] though with some changes in notation. A very helpful feature is the careful definition of all tabulated quantities as well as a description of their physical significance. The principal quantities

tabulated are the so-called *intensity functions* i_1 and i_2,

$$i_1 = |i_1^*|^2 = \left| \sum_{n=1}^{\infty} \{A_n\pi_n + P_n[x\pi_n - (1 - x^2)\pi_n']\} \right|^2,$$

$$i_2 = |i_2^*|^2 = \left| \sum_{n=1}^{\infty} \{A_n[x\pi_n - (1 - x^2)\pi_n'] + P_n\pi_n\} \right|^2,$$

where $\pi_n(x) = \partial P_n(x)/\partial x$, and $\pi_n'(x) = \partial^2 P_n(x)/\partial x^2$, $P_n(x)$ being the Legendre polynomial of degree n. $A_n = a_n/n(n+1)$, $P_n = p_n/n(n+1)$, a_n and p_n being complicated expressions involving Bessel functions of half-integral order. i_1 and i_2 give the angular distribution of intensity and the total radiation scattered by a small spherical particle as a function of $\alpha = 2\pi r/\lambda$, where r is the radius of the particle and λ is the wave length of the radiation. Among various tables of **Part I** (p. 1–51) i_1 and i_2 are tabulated for values of the real index of refraction m of the scattering particle equal to 1.33 (that for water), 1.44, 1.55, and 2, and for values of $\alpha = .5, .6, 1, 1.2, 1.5, 1.8, 2, 2.4, 2.5, 3, 3.6, 4, 4.8, 5, 6$, as the angle of scattering γ (the angle between the direction of propagation of the scattered radiation and the reversed direction of propagation of the incident radiation) ranges 0(10°)180°. The other functions tabulated are real and imaginary parts of (i) A_n, P_n, for various values of n, (ii) i_1, i_2, as well as values of $\frac{1}{2}\alpha^2 K(m;\alpha) = \frac{1}{2}\int_0^\pi (i_1 + i_2) \sin\gamma d\gamma$, where K represents the total scattering coefficient or the total energy scattered per second per unit cross-sectional area of particle, illuminated by radiation of unit intensity.

The scattering of light by colloidal suspensions offers a useful method for the determination of the size of the scattering particles. The present tables afford a means of estimating this from the experimentally observed intensity and scattering coefficient functions.

In **Part II**, p. 53–59, three functions are tabulated, for a transparent particle of refractive index 1.5, for a range of α from .5 to 12, and for pairs of values whose ratio is 1.2, which is the wave length ratio of the two most distinctive colors observed, namely: red light ($\lambda = 6290$ A°) and green light ($\lambda = 5240$ A°). These functions are $K(m;\alpha)$, $(2n+1)R(C_n^1)$, $(2n+1)R(C_n^2)$, where $C_n^1 = (-1)^{n+1}ia_n/(2n+1)$; $C_n^2 = (-1)^n ip_n/(2n+1)$.

In the tables of **Part III** (p. 61–81) are presented the values of

$$F(m;\alpha) = (2/\alpha^2) \sum_{n=1}^{\infty} (2n+1)\{C_n^1(m;\alpha) + C_n^2(m;\alpha)\} = K(m;\alpha) + iL(m;\alpha),$$

for a medium containing *absorbing* particles with extinction coefficient k varying from 0 to .1. This quantity is defined in such a way that $e^{-4\pi k}$ is the fraction of the radiation absorbed in travelling a distance λ through the bulk material. The tables here are restricted to the range $m = [1.44(.01)1.55$; 4S or 5D] for the real part of the complex index of refraction.

The dispersion and absorption of electromagnetic radiation in liquid water are also of importance in the microwave radar region of wave lengths from 10 centimeters to 3 millimeters. Consequently in **Part IV** (p. 83–119), tables are included giving the real and imaginary parts of C_n^1 and C_n^2, 4D, for significant values of n, and the scattering function $K(m^*;\alpha)$, 3D, for complex indices of refraction m^* corresponding to this wave length interval. The tabulations are for the following special cases of m^*: $4.21 - 2.51i$, $\alpha = .1(.05)1(.1)3$; $5.55 - 2.85i$, $\alpha = .1(.05)1(.1)2$; $8.18 - 1.96i$, $\alpha = .1(.025)1$; $3.41 - 1.94i$, $\alpha = .1(.05)1(.1)5$; $7.20 - 2.65i$, $\alpha = .1(.025)1(.05)1.3$; $8.90 - 0.69i$, $\alpha = .1(.01).3(.005).43(.01).6$.

This collection of tables will be of considerable value to investigators in the application of visible light scattering to the study of suspensions as well as to those who study microwave scattering. It is to be hoped that similar tables may ultimately be prepared for the scattering of sound by spherical obstacles.

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[1] G. MIE, "Beiträge zur Optik trüber Medien," *Annalen d. Physik*, s. 4, v. 25, 1908, p. 377–445.

MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in RMT **633** (Boll), **646** (Newman, Powell).

156.—H. W. Holtappel, *Tafels van e^x*, Groningen, 1938. See *MTAC*, v. 1, p. 437–438; 449–451.

In addition to the long list of errors already reported in *MTAC* by NBSCL, we note the following in the table of e^x: $x = 6.450$, *for* 632.70229 28133, *read* 632.70229 28123.

NBSINA

Los Angeles

157.—T. L. Kelley, *The Kelley Statistical Tables*. New York, 1938, *MTAC*, v. 1, p. 151–152; Revised ed., Cambridge, Mass., 1948, *MTAC*, v. 3, p. 301–302.

The following errors are to be found in both editions of the table of $(1 - p^2)^{\frac{1}{2}}$—the first page reference is to the 1938 edition, and the second to the 1948 edition:

Page	p	*For*	*Read*	Page	p	*For*	*Read*
25(49)	.5581	.8297 8370	.8297 7370	25(49)	.5587	.8293 7983	.8293 6983
	.5582	.8297 1643	.8297 0643	38(62)	.6208	.7839 7898	.7839 6898
	.5583	.8296 4914	.8296 3914		.6209	.7838 9978	.7838 8978
	.5584	.8295 8184	.8295 7184	60(84)	.7344	.6788 1691	.6787 1691
	.5585	.8295 1452	.8295 0452	71(95)	.7874	.6164 3241	.6164 4241
	.5586	.8294 4718	.8294 3718				

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158.—K. Pearson, *Tables of the Incomplete Beta-Function*. Cambridge, 1934.

On p. XXXV, footnote referring to *Tracts for Computers*, VIII, *Table of the Logarithms of the Complete* Γ*-Function*, to 10D, it is stated that it was for $p = 2$ to 1200, "argument intervals 0.5, 1, and 2." *For this read:* $p = 2(.1)5(.2)70(1)1200$.

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159.—U. S. Coast and Geodetic Survey, *Natural Sines and Cosines to Eight Decimal Places*, 1942; reprinted with some corrections in 1946.

Editorial Note: In **MTE 155 A**, p. 424, it was stated that all of our previously published errata in this volume had been corrected in a new printing. We regret that in this respect we misrepresented Prof. E. G. H. Comfort's previous report. He has quite rightly pointed out that 9 of our 12 earlier reported (before 1949) errors still remain in the reprint, namely: the 6, v. 1, p. 65, and the 3, v. 1, p. 87.

UNPUBLISHED MATHEMATICAL TABLES

79[B].—Bartol Research Foundation, Swarthmore, Pa. At this Foundation tables of the functions $(1 - x^2)^{-1}$ and $(1 - x^2)^{-\frac{1}{2}}$ have been calculated for $x = [0(.0001).99; 6D]$. There is an uncertainty of one unit in the sixth decimal place.

Stephen H. Forbes

80[F].—A. GLODEN, *Nouvelle extension des solutions de la congruence* $x^4 + 1 \equiv 0$ $(mod\ p)$ *pour* p *entre* $6 \cdot 10^5$ *et* 10^7. Mss. in possession of the author, 11 rue Jean Jaurès, Luxembourg, and in the Library of Brown University.

This manuscript table gives solutions x of the congruence mentioned in the title for approximately 1300 primes p beyond the limit 600000 set by previous tables [1] and under ten millions. With a few exceptions, only one solution x is given for each p (instead of the usual pair) and in each $x < 40000$. The table is a byproduct of the results of factoring numbers of the form $x^4 + 1$.

D. H. L.

[1] For references to previous tables of this sort see *MTAC* v. 1, p. 6, v. 2, p. 71, 210, v. 3, p. 96.

81[F].—A. GLODEN, *Table de factorisations des nombres* $N^8 + 1$ *pour* $N \leq 400$. Mss. in possession of the author, 11 rue Jean Jaurès, Luxembourg, and in the Library of Brown University.

The table of CUNNINGHAM [1] giving the factors of $N^8 + 1$ for $N \leq 200$ is extended in this manuscript not only by doubling the upper limit of N but also by raising limit 100000 of the smallest prime factor omitted to 600000. Of the 400 entries of the table 140 are complete factorizations, 213 are composite but incompletely factored, while only 47 are of entirely unknown character, beyond the fact that their factors lie above 600000. The smallest number of this latter kind is $\frac{1}{2}(43^8 + 1) = 5844100138801$. The author hopes to raise his 600000 to 800000 by extending his already extensive tables of solutions of the quartic congruence[2] $x^4 + 1 \equiv 0$ (mod p).

A comparison of this table with that of CUNNINGHAM reveals the following errata in the latter.

p. 140	$y = 86$	insert the small factor 61057
	$y = 148$	delete the factor 97
p. 141	$y = 125$	for semicolon read full stop.

D. H. L.

[1] A. J. C. CUNNINGHAM, *Binomial Factorisations*, v. 1, London 1923, p. 140–141.
[2] See *MTAC*, v. 2, p. 71–72, 210–211, 252, 300.

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 418 South Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

Our contribution under this heading, appearing earlier in this issue, is "The solution of simultaneous linear equations with the aid of the 602 calculating punch," by FRANK M. VERZUH.

DISCUSSIONS

A New General Method for Finding Roots of Polynomial Equations

The problem of finding all of the roots of polynomial equations of fairly high degree arises so frequently that a routine for accomplishing this automatically on a high-speed digital computer would be of considerable practical value. However, for every one of the standard methods for finding the roots of a polynomial equation, there are some exceptional cases in which the particular method applied fails to work.

Horner's method and Newton's method require a good initial approximation, to prevent

the resulting sequence of approximations from diverging (as in the example $f(x) = x^3 - 5x$, where $x_0 = 1$) or involving division by zero. The method of false position will fail completely for complex roots and for real roots of even multiplicity. Both Lagrange's method and the method of Sturm functions fail for complex roots. The Graeffe method will furnish the absolute value of each root and the total number of roots having each absolute value, but it will not give the roots themselves if there are several complex roots of equal modulus. Since in the case of real polynomials the complex roots always occur in pairs of equal modulus, further refinements of the method are necessary.[1] The refined methods either fail or become extremely complicated in the case of several pairs of roots of equal modulus. In using the Graeffe method efficiently, it is also necessary to exercise considerable judgment to know which one of the possible modifications to use or to know when to stop the squaring process. Bernoulli's method does not converge if there is more than one root of largest absolute value. A modification of Bernoulli's method [2] will give the absolute value if there are n roots having this absolute value, but a different method is required for each different n. Also, considerable judgment must be exercised to know which method to use or when to change methods.

In the case of a mathematician working with a desk computer, exceptional cases will not cause much trouble, since they occur rarely, and, by the use of intelligent judgment, they can be detected and an appropriate change of method made. But it would be an imposing task (even without considering the limitations necessitated by the proposed memory sizes of all the digital machines now under construction) to set up routines for detecting and handling every one of the possible combinations of exceptional cases that can arise. It would be of advantage to have a single method which will always give all of the roots, regardless of their positions or multiplicities.

The method outlined below is a first attempt at such a universal method. It is admittedly not a speedy or efficient one, since, in fact, it would require a prohibitive amount of time to carry out the method except on high-speed digital calculating machinery, and even on these machines the process would be time consuming. But it has the advantage that regardless of the nature of the original polynomial, this method will always converge to a root, to as much accuracy as the machine uses. Furthermore, the convergence always takes place within a fixed number of steps (independent of the degree of the polynomial equation).

After finding the "first" root, the equation can be reduced in degree by removal of this root and the other roots found by repeating the same routine. Thus, the entire set of roots and their multiplicities can be found by a machine without the necessity of any human intervention during the problem.

The proposed method is as follows: Given any point p_n which is the nth approximation to a root, the following operations produce the $(n + 1)^{\text{th}}$ approximation. Expand the polynomial around the point p_n by synthetic division. Then, by performing the Graeffe process a fixed number of times, find (to some known degree of accuracy in terms of relative error) R, the absolute value of the root of the transformed equation which is smallest in modulus. Since R is a function of p_n, it will be denoted by $R(p_n)$, and geometrically it is the distance from the point p_n to the nearest root of the original polynomial.

The number actually obtained by applying the Graeffe root-squaring process a fixed number of times will be denoted by $R^*(p_n)$, where $(1 + d_n)R^*(p_n) = R(p_n)$, and by using enough significant figures in the synthetic division and the Graeffe process it is possible to insure that the error term d_n satisfies $|d_n| << 1$.

Since there will be at least one root of the polynomial equation near the circumference C_n of the circle of radius $R^*(p_n)$ about the center p_n, if a suitable set, S_n, of points (such as the vertices of a regular inscribed heptagon or octagon [3]) on C_n are chosen, S_n will contain at least one point s such that $R^*(s) \leq \frac{1}{2}R^*(p_n)$. Choose any one, s (e.g., the first one which is tried) of these points, and call it p_{n+1}.

Starting with the initial approximation $p_0 = 0$ and finding each successive p_n by iterative applications of the above procedure, one can easily verify by induction from $R^*(p_{n+1}) \leq \frac{1}{2}R^*(p_n)$ that $R^*(p_n) \leq 2^{-n}R^*(p_0)$.

Then, if p denotes a root of the original equation which is nearest to p_0, it follows that $|p| \geq R(p_0) = (1 + d_0)R^*(p_0)$.

The relative error of p_n, considered as an approximation to p, is

$$\frac{|p_n - p|}{|p|} = \frac{R(p_n)}{|p|} \leq \frac{(1 + d_n)R^*(p_n)}{(1 + d_0)R^*(p_0)} \leq \frac{(1 + d_n)2^{-n}R^*(p_0)}{(1 + d_0)R^*(p_0)} = \frac{1 + d_n}{1 + d_0} 2^{-n} \approx 2^{-n}.$$

It is clear that on a binary computer the nth approximation p_n is good to about n significant figures, if the computer carries several extra significant figures in the calculation to assure that $|d_n|$ is always sufficiently small.

This method is universal in that it does not depend on the location or multiplicities of the roots or on the degree of the original equation.

Since the Graeffe process easily gives the number of roots having the absolute value R, the multiplicity of the root to which p_n converges would be apparent without further calculation.

Complex numbers can arise in the course of the calculation, even when the original coefficients are real numbers. For this reason, and also because very large powers of 2 or 10 can arise as exponents, the use of corresponding complex, floating-point addition and multiplication computer subroutines will be necessary. Such subroutines make it possible for the original equation to have complex coefficients covering a very wide range of magnitudes without involving extra programing.

The time required to find one root by this method (as calculated from the number of multiplications and additions) will be proportional to the square of the degree of the original polynomial. It follows that the time required to find all roots will be proportional to the cube of the degree of the polynomial.

Several modifications of this method could be made which would speed up convergence, but probably not sufficiently to make the method actually feasible for extensive machine use. The constant $\frac{1}{2}$, the number of points in S_n, and the degree of accuracy to which R should be calculated, could be adjusted to minimize the time. Also, it would save time to test for convergence during the computation rather than to iterate a fixed number of times.

The author wishes to express his appreciation to Dr. L. B. Tuckerman, Jr., Dr. E. W. Cannon, & Mr. George Gourrich, all of the NBS, for their suggestions and assistance in developing this method.

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[1] E. Bodewig, "On Graeffe's method for solving algebraic equations," *Quart. Appl. Math.*, v. 4, 1946, p. 177–190.

[2] Bernard Dimsdale, "On Bernoulli's method for solving algebraic equations," *Quart. Appl. Math.*, v. 6, 1948, p. 77–81.

[3] The choice of an octagon will allow enough overlap to permit this method to converge even if $|d_n|$ is fairly large.

Bibliography Z-VIII

1. Andrew D. Booth & Kathleen H. V. Britten, *Coding for ARC*, [Automatic Relay Computer]. Second edition, July 1948, 39 leaves. British Rubber Producers' Research Association, Welwyn Garden City, Herts., England. 20.3 × 25.4 cm. Not available for distribution.
2. Andrew D. Booth & Kathleen H. V. Britten, *General Considerations in the Design of an All-Purpose Electronic Digital Computer*. Second edition, Aug. 1947, 24 leaves, mimeographed. British Rubber Producers' Research Association, Welwyn Garden City. 21.6 × 27.9 cm. Not available for distribution. Copies of nos. 1 and 2 may be consulted in the Library of The Institute for Advanced Study.
3. Arnold A. Cohen & William R. Keye, "Selective alteration of digital data in a magnetic drum computer memory," transcript of a paper presented at the Institute of Radio Engineers, National Convention, May 1,

1948; available from Engineering Research Associates, Inc., St. Paul, Minn., 10 p., illustr. 27.5 × 21.4 cm.

4. J. M. Coombs, "Storage of numbers on magnetic tape," National Electronics Conference, *Proc.*, v. 3, 1947, p. 201–209. 15.3 × 22.7 cm.

Abstract: This paper describes a memory system for storing digital information on magnetic tapes. The tapes are bonded to the surface of an aluminum drum. Associated with each tape are three heads for reading, writing, and erasing magnetized spots on the tapes. This equipment allows numbers to be stored indefinitely, to be inspected as often as required, and to be removed when no longer needed. The system will store 200,000 magnetized spots on a drum 34 inches in diameter and 10 inches wide.

5. Andrew Hamilton, "Brains that click," *Popular Mechanics Mag.*, Mar. 1949, p. 162–167, 256, 258, illustrs. 22.5 × 24.1 cm.

6. G. G. Hoberg and J. N. Ulman, Jr., *Glossary of Computer Terms*, Special Devices Center, Office of Naval Research, Port Washington, Long Island, N. Y., May 1948, 12 p. 21.6 × 27.9 cm.

From the Foreword: The reporter of a new art is always confronted with the problem of describing previously unknown concepts and devices for which there are no words in our general vocabulary. Project Whirlwind reports inevitably contain a considerable number of specialized terms used in new senses. This Glossary has been prepared for distribution to the recipients of these reports in the hope that it will clarify any terms whose meanings might be strange or in doubt.

7. C. H. Page, "Digital computer switching circuits," *Electronics*, v. 21, 1948, p. 110–118. 21.6 × 27.9 cm.

Basic operational requirements of digital computers and fundamentals of the means for obtaining them are set forth. For the most part familiar switching circuits can be used, but they must meet the special requirements of positive action described in this paper.

8. G. W. Patterson, R. L. Snyder, L. P. Tabor, & Irven Travis, *The EDVAC, a Preliminary Report on Logic and Design*, Moore School of Electrical Engineering, University of Pennsylvania, Feb. 16, 1948, iv, 100 leaves; 3 folding plates in pocket. 21.6 × 27.4 cm.

Two reports on the EDVAC [Electronic Discrete Variable Computer] dated September 30, 1945, and June 30, 1946 [see *MTAC*, v. 3, p. 379–380], discussed at length the relative merits of various projected machines, circuits, and computer techniques but did not describe in detail any one machine plan. Since then the plan of the EDVAC has been almost entirely frozen, and the construction is far advanced. This report is mostly devoted to a careful description of exactly how orders and numbers are represented in the EDVAC, how it treats them, and what basic operations it can perform. Therefore, this report is required reading for anyone who expects to prepare problems for the EDVAC. Considerations dictating particular choices in logical design are seldom weighed.

After a three-page historical introduction, the general organization of the EDVAC is explained in fifteen pages supplemented by a system block diagram. The reader need not be familiar with the EDVAC or with prior reports on it in order to understand the report under review. Seven pages deal cursorily with design problems, the real purpose being to give the unfamiliar reader a picture of the EDVAC's physical structure.

Mathematicians are familiar with the important logical distinction between a set containing only one element and the element itself. Digital computers abound in just such distinct objects which stand in "one-to-one" correspondence and are confused in common discourse. Consider for example: (1) a sequence of 44 characters (pulses or spaces) which

the machine interprets as an order, (2) the 10-character segment of the order called the second address, (3) the number into which this address itself is interpreted, (4) the position in the memory denoted by this address, (5) the sequence of 44 characters stored in this memory position and called a word, and (6) the number into which this word is interpreted for arithmetic operations. This report introduces an elaborate symbolic functional notation to make such distinctions compactly and unambiguously. Letter symbols are chosen with mnemonic values so that the reviewer was able to follow the descriptions quite easily; however, he has been intimately engaged in planning the NBS Interim Computer which is very similar to the EDVAC in logical design. Readers less familiar with this design complained that the use made of the notation does not justify the effort of learning it. Perhaps a discussion less "scrupulously accurate" and containing more "clumsy locutions" would be easier for beginners; however, the reviewer approves the introduction of the symbolism, because he feels that a compact precise language for exact communication between specialists is needed. Furthermore he hopes that ultimately both the invention and checking of logical designs of computing machines will be facilitated by a suitably formalized logical calculus. The report should, however, have included a list of terminology and symbols for ready location of forgotten definitions.

A large drawing shows the control panel of the EDVAC. Its use is described in six pages. A bare introduction to the reading of logical block diagrams at the level where tubes, gates, flipflops, and delay lines are shown as blocks is given in seven pages which trace through just that part of the computer which performs addition and subtraction without checking. The final pages of the report treat briefly the diagnosis of errors and give design suggestions for future machines of the EDVAC type.

This report was written primarily for persons who must prepare problems for the EDVAC. It gives the data which are indispensable to them, namely, what the EDVAC will do in response to any possible coded order. The engineering discussions are calculated to supply background and to satisfy the curiosity of these people whose training is more in mathematics than in engineering. It would be a grave error to employ these oversimplified arguments as the bases of important engineering decisions.

9. JOHN PFEIFFER, "The machine that plays gin rummy," *Science Illustrated*, v. 4, Mar. 1949, p. 46–48, 84, 86–88, illustr. 27.9 × 20.3 cm. See also "Machine plays chess," *Science News Letter*, v. 54, 21 Aug. 1948, p. 123.

10. M. V. WILKES & W. RENWICK, "An ultrasonic memory unit for the EDSAC," [Electronic Delay Storage Automatic Calculator], *Electronic Engin.*, London, v. 20, 1948, p. 208–213, illustr. 19 × 25.4 cm.

11. M. V. WILKES, "The design of a practical high-speed computing machine. The EDSAC," [Electronic Delay Storage Automatic Calculator], R. Soc. London, *Proc.*, v. 195 A, 1948, p. 274–279. 17.2 × 25.4 cm.

NEWS

Association for Computing Machinery.—The balloting resulted in the election of the Council of the Association as listed in *MTAC*, v. 3, p. 380. The Council, according to the provisional Constitution and Bylaws, holds office until May 31, 1949. The new Council met in New York on Jan. 7. Messrs. E. C. BERKELEY and R. V. D. CAMPBELL were formally elected to serve as secretary and treasurer, respectively, until May 31, 1950.

At present, a committee under the chairmanship of Mr. E. G. ANDREWS is compiling a nomenclature list for large-scale computers. The list is to include items relating to: (1) general computer terminology (i.e., terms relating to digital computers, special-purpose computers, etc.); (2) computer components (i.e., storage, arithmetical organ, etc.); (3) notation (modified binary systems, etc.); and (4) operation (i.e., programing, etc.).

Institute of Radio Engineers.—The program of the National Convention of the IRE, March 7–10, 1949, New York City, included, on March 8, two sessions concerned with large-scale computers. The first session, on *Electronic Computers*, E. W. Cannon, chairman, included the following papers: "A dynamically regenerated memory tube," by J. P. Eckert, Jr., H. Lukoff, & G. Smoliar, Eckert-Mauchly Computer Corp., Philadelphia; "An electronic differential analyzer," by Alan B. MacNee, Massachusetts Institute of Technology; "An analogue computer for the solution of linear simultaneous equations," by Robert M. Walker, Watson Scientific Computing Laboratory; "The electronic isograph for a rapid analogue solution of algebraic equations," by Byron O. Marshall, Jr., Cambridge Field Station, Air Matériel Command; and "Parametric electronic computer," by Charles J. Hirsch, Hazeltine Electronics Corporation, Little Neck, L. I.

In the first of these papers a description was presented of the work of the Eckert-Mauchly Computer Corp., and of the University of Manchester group under F. C. Williams, on the use of a nearly standard cathode-ray tube to provide storage of a large number of binary digits at a relatively low cost per digit, and more particularly with an access time of not more than a few microseconds. The system is advantageous in that it embodies the ability to move from one part of the memory to any other at high speed. Secondly, the low cost and easy maintenance allowed by the use of completely standard high-production electronic components with no special tubes make it an extremely desirable high-speed memory system. A detailed analysis of a computer based on such a system is now possible, and, in this connection, models of the memory system are being set up to further assess its reliability and practicality.

Mr. MacNee's talk concerned an electronic differential analyzer, capable of solving up to sixth order ordinary differential equations, both linear and nonlinear. Its high operation speed and extreme flexibility permit rapid investigation of wide ranges of equation solutions with regard to periodicity, instability, and discontinuities. Two new computing elements, an electronic function generator and an electronic multiplier, are employed.

The electrical analogue computer described by Mr. Walker accepts information for systems of linear equations of up to 12 unknowns in digital form from a set of punched cards thus facilitating the preparation, checking, and insertion of input data. Solutions of well-determined problems are easily and rapidly obtained, and may be refined to any desired accuracy by a simple iteration procedure.

Following this was a discussion of an electronic analogue computer known as an isograph, with which the complex plane may be rapidly investigated for roots of up to 10th-degree polynomials. The value of any polynomial is given for all values of the complex variable. It is believed that the isograph, which may be used alone or in conjunction with a large-scale computer, will be of special value in servo-mechanisms and that, in general, theoretical analyses of engineering problems will be furthered by its use.

The last talk dealt with a novel computer that operates on the principle of an alignment chart wherein data voltages are aligned in time in the same manner that data quantities are aligned in distance on a slide rule. Since the "time scales" of this electronic slide rule may be calibrated according to any function of time which can be electrically realized, a large variety of operations can be performed. Because of the rapidity of operation, they can be repetitive and performed on variable parameters. This computer can be made as accurate as desired if high enough voltages and rapid enough samplings are taken.

The second session, on *Electronic Computing Machines*, Dr. E. U. Condon, chairman, included the following papers: "Results of tests on the BINAC," by J. W. Mauchly, Eckert-Mauchly Computer Corp.; "The Mark III computer," by H. H. Aiken, Harvard University; "The IBM type 604 electronic calculator," by Ralph Palmer, IBM, New York; "Electrostatic memory for a binary computer," by F. C. Williams, University of Manchester, England; "Counting computers," by G. R. Stibitz, Burlington, Vt.; and "Programing a computer for playing chess," by Claude E. Shannon, Bell Telephone Laboratories, New York.

The operating characteristics of the BINAC, which represents the practical application of computer elements discussed at last year's convention (i.e., a mercury delay-line memory

and magnetic tape input and output equipment), were discussed by Dr. Mauchly. This device is of particular interest because, except for input and output equipment, it is entirely electronic.

The Mark III, the latest of the large-scale computers developed under Prof. Aiken's direction, has been designed for greater speed and reliability, more flexible memory facilities, and greater ease of preparation of input data than were found in the earlier computers.

The type 604 electronic calculator described by Mr. Palmer combines an electronic arithmetic element, including a 13-digit electronic counter, with punched-card input and output equipment and additional mechanical storage registers, with the possibility of carrying out automatically a "program" of as many as 20 arithmetic operations.

Dr. Williams treated the electrostatic memory on which he has carried out extensive research. These memories probably show the greatest promise of any of the basic types thus far proposed for computers, since they combine the high reading and writing speed of the delay-line type of memory with a very short "access time."

Dr. Stibitz proposed a new type of computer which combines the more accurate elements of the familiar differential analyzer, such as gears and differentials, with a new type of "function unit," resulting in a computer having the simplicity and low cost of an analogue computer and the higher accuracy of the digital type.

Dr. Shannon discussed the programing of a chess game on a large-scale computer. While the possibility of such applications was early recognized, this probably represents the first serious attempt to analyze the programing of such an operation.

Massachusetts Institute of Technology.—A Special Course in Analogue Computation, designed particularly to meet the needs of users of industrial types of analogue computing machines, was initiated at the Massachusetts Institute of Technology on June 20, 1949, to last for three weeks. The course was presented by Dr. SAMUEL H. CALDWELL, professor of electrical engineering and director of the Institute's Center of Analysis, and dealt especially with the treatment of engineering problems by machines designed for the solution of differential equations. The objective of the course was to provide a broader understanding of the uses and potentialities of analogue computers. The increasing availability of these machines throughout industry makes it important that trained personnel be prepared fully to exploit their benefits. Demonstrations were arranged using the MIT Differential Analyzer, as well as various types of electronic differential analyzers available or under development at the Institute. The course included a unified treatment of the following subject matter: mathematics refresher, basic analogue processes, machine solution of differential equations, calculation of scale factors, and electronic analogue machines.

Swedish State Board for Computing Machinery.—The Swedish State Board for Computing Machinery has recently been formed with Admiral STIG ERICSSON as president and Professors T. LAURENT, E. VELANDER, N. ZEILON (of Lund), and permanent secretary G. A. WIDELL (of Stockholm) as members. The Board's secretary is GÖSTA MALMBERG (Ecklesiastikdepartementet, Stockholm).

The immediate plans for the future include the construction of a relay computer in agreement with a project by Dr. CONNY PALM at Stockholm Institute of Technology. No definite plans exist regarding the design and construction of an electronic computer.

Any inquiries should be addressed to the secretary, or to Dr. C. E. FRÖBERG, Institute of Mechanics and Mathematical Physics, Lund.

OTHER AIDS TO COMPUTATION

BIBLIOGRAPHY Z–VIII

1. SIDNEY G. HACKER, *Arithmetical View Points. An Introduction to Mathematical Thinking*. Pullman, Wash., State College of Washington Bookstore, 1948, x, 144 p. + 14 plates. 21 × 27.4 cm. $1.50. Offset print.

This interesting *potpourri* by a professor of mathematics represents a course of lectures, which might be read with interest by senior undergraduates. There are 6 main headings.

I., p. 3–14, The Art of Reckoning. II., p. 15–46, Mechanical Counting Devices: The subheadings are 1. The Abacus, 2. The semi-automatic desk computing machine; 3. Charles Babbage "engines"; 4. The Harvard IBM automatic sequence controlled calculator (ASCC); 5. The ENIAC; 6. Relative merits of the ASCC and the ENIAC; 7. The differential analyzer. Digital and analogue machines; 8. Establishment of national mathematical laboratories in the United States and Europe; 9. The need for students of numerical analysis. III., p. 47–64, Certain of the Foundations of Arithmetic. IV., p. 65–84. V., p. 85–104, The Rational Numbers. VI., p. 105–123, The Irrational Numbers. P. 126–143 annotated list of literature references. The illustrations admirably reproduced are: The Peruvian knotted calculating cards; Japanese soroban; one of Pascal's computing machines; Leibniz's calculator; Calculating wheels of Babbage's "difference engine"; The mill and printing parts of Babbage's "analytical engine"; The Harvard ASCC, Mark I (4 views); The ASCC, Mark II (Naval Proving Ground, Dahlgren, Va.), (2 views); Graphs in connection with Fermat's last theorem, $x^n + y^n = 1$, $n = 1(1)5$, for positive x and y; A general view of the ENIAC.

A mimeographed errata sheet with 25 entries accompanies the volume.

R. C. A.

2. Edmond R. Kiely, *Surveying Instruments Their History and Classroom Use.* (National Council of Teachers of Mathematics, *Nineteenth Yearbook.*) New York, Columbia Univ., Teachers College, 1947, xvi, 411 p. 15 × 22.8 cm.

Contents: Beginnings in Egypt, China, and Babylonia (p. 1–17); Developments in Greece and Rome (p. 18–44); Contributions of Medieval Europe, Islam and India (p. 45–100); Advancements in Europe during the Renaissance (p. 101–238); Development of practical geometry in the schools (p. 239–263); Applications of geometry, trigonometry in simple surveying (p. 264–360); Bibliography (p. 377–396) 557 titles; Index (p. 397–411). 270 illustrations and figures.

3. Fritz Reinhardt, "Der logarithmische Rechenzylinder für komplexe Zahlen," *ETZ, Elektrotech. Z.*, v. 69, 1948, p. 78–82.

Translated Summary: "By means of the analytic function $w = \ln z$ the Gaussian plane of complex numbers z is conformally mapped on a strip of the w-plane. Thus the polar coordinates A, α of a vector $\mathbf{A} = A \cdot e^{i\alpha} = a_1 + ia_2$, go over into rectangular coordinates of its image in the w-plane, while the rectangular coordinates of the vector with components a_1 and a_2, transform into the parameter values of a family of orthogonal curves. Logarithmic laws apply to complex numbers; consequently the representation of the product or quotient of two plane vectors in the z-plane, is a sum or difference in the w-plane. The image curves in the w-plane repeat themselves periodically in the direction of the ordinate for each 90°, and in the abscissa-direction for every power of 10. Hence it is possible to roll the image plane on a cylinder. Over this is slipped a transparent cylinder with indicators and this cylinder again carries another short movable transparent cylinder with one reading mark. This device allows us to represent every vector and to add it to, or to subtract it from, a second vector. Thus one obtains a logarithmic computing cylinder for complex numbers by means of which we can carry out, not only computations for components, but also multiplication and division of complex numbers similarly to the procedure of an ordinary slide rule for real numbers. The result can be found either in rectangular or in polar coordinates as one may desire.

4[H, I, Z].—F. A. Willers, *Practical Analysis, graphical and numerical Methods*. Translated by R. T. Beyer. New York, Dover Publications, 1948, x, 422 p. 15.2 × 23.5 cm. $6.00.

This is a translation of the German edition, *Methoden der praktischen Analysis*. Berlin, 1928, modified in only two sections concerned with the slide rule and desk calculators.

The work is in 6 chapters entitled I. Numerical Calculation and its Aids, II. Interpolation, III. Approximate Integration and Differentiation, IV. Practical Equation Theory, V. Analysis of Empirical Functions, VI. Approximate Integration of Ordinary Differential Equations.

In simply turning over the pages one is struck with the large number of figures. The book contains as many as 132 figures, an average of 1 to every three pages. This is partly due to the fact that there is more space than usual devoted to graphical methods and computational instruments. In the author's foreword we find: "I still believe it necessary to describe the graphical methods, since I am of the opinion that they are of practical importance." Whether or not the reader feels that this opinion has been shaken by the last two decades he will find the treatment of graphical and instrumental subjects very complete and well done. The account of the planimeter is especially good.

There is much space devoted to definite examples. Not only is the numerical work shown in great detail (indicating that in many cases the computer is using merely paper and pencil) but also the numerical problem is often set up *ab initio* from a physical situation.

Each chapter has about 6 sections. Of the 35 sections of the book over such a wide range of topics it is impossible to discuss each one here. They are for the most part unconnected so that a reader may use the book as a sort of an encyclopaedia. Many of the sections are necessarily too short to give a complete account of the particular subject. The author has wisely preferred to illustrate the fundamental ideas and to leave the reader a short bibliography collected at the end of each section with which further to pursue the topic. Here the reader who is unfamiliar with German will be disappointed, since nearly all the references are to works in German. It would have been very useful had the translator supplied additional if not alternative data in English. Also many of the references are out of date, all being at least 20 years old.

Section 3 on the slide rule was rewritten by the translator to deal with American type rules. No mention is made however of the existence of the circular type rules.

Section 6 is written by T. W. SIMPSON and gives a fine account of the three standard American desk calculators and the two dozen different operational techniques not described in the manufacturers' booklets. As might be expected, there are difficulties in nomenclature not encountered elsewhere in the book. These are largely overcome by careful writing. This section contains the only real tables in the book. These are two tables facilitating square and cube rooting. For description of these tables see *MTAC*, v. 1, p. 356–357, v. 2, p. 350–351.

The printing is excellent with the exception of several of the line drawings which have been poorly reproduced.

This book should find its way into many a computing room. Its use as a text in an upper division college course in numerical methods is also clearly indicated. Many of its sections like §16 (Mean value Methods) are excellent lecture material. Others like §22 (GRAEFFE's Method) could be easily amplified to whatever extent the instructor desired.

D. H. L.

NOTES

103. ARNOLD NOAH LOWAN.—In *Scripta Mathematica*, v. 15, p. 33–63, March, 1949, Dr. Lowan has recently presented a detailed account of the work of the Computation Laboratory, which since its foundation in January 1938 has been under the sponsorship of the National Bureau of Standards. During all of this time Dr. Lowan has been the director of the technical planning of this group. Since, after recent reorganization by the NBS, only a very few computers or planners are left in the CL, Dr. Lowan's extraordinarily successful period as director has this month been brought to a close. The wisdom of the NBS procedure in this regard may be doubted.

The great publication output of members of the NBSCL and preceding organizations, 1939–1949, is listed in the article referred to above, and a

partial list of those tables with the preparation of which Dr. Lowan was more or less directly connected, appears in R. C. ARCHIBALD, *Mathematical Table Makers*, New York, 1948.

Dr. Lowan was born in Jassy, Roumania, in 1898. He graduated as chemical engineer at the Polytechnic Institute of Bucharest in 1924, the year that he arrived in America; in 1929 he became a naturalized American citizen. During 1928–1931 he was a research physicist for the Combustion Utilities Corp., Linden, N. J., and received the M.Sc. degree from New York University in 1929. His Ph.D. degree was granted by Columbia University in 1934.

As a recognition of Dr. Lowan's notable scientific services we take pleasure in presenting his portrait as our frontispiece for this issue.

EDITORS

104. ROOTS OF CERTAIN TRANSCENDENTAL EQUATIONS.—The FMR, *Index* does not indicate any existing tables of the roots of the equations $\tan x + x = 0$ and $\tan x + 2x = 0$.

The functions arise in a study of the extreme values of

$$f(x) = x \sin x.$$

It is evident that a solution may be obtained with a slight modification of the method developed by EULER [1] and independently by Lord RAYLEIGH.[2] It is assumed that

$$x = (n + \tfrac{1}{2})\pi + y = \phi + y \quad (n = 0, 1, 2, \cdots),$$

where y is a positive quantity which is small when x is large. Then

$$\tan y = (\phi + y)^{-1}$$

and

$$y = \phi^{-1} - \phi^{-2}y + \phi^{-3}y^2 - \cdots - \frac{1}{3}y^3 - \frac{2}{15}y^5 - \frac{17}{315}y^7 - \cdots.$$

Solving this equation by successive approximation we obtain

$$x = \phi + \phi^{-1} - \frac{4}{3}\phi^{-3} + \frac{53}{15}\phi^{-5} - \frac{1226}{105}\phi^{-7} + \frac{13597}{315}\phi^{-9} - \cdots, \tag{1}$$

where $\phi = (n + \frac{1}{2})\pi$.

When $n < 2$, this equation is not suitable for computation, because of slow convergence. The first two roots can best be determined by the use of trigonometric tables. For the higher roots the variation of the tabular values of the tangent become so rapid that use of the series expansion is preferable and the convergence of the series increases rapidly with increasing n.

For the second equation, $\tan x + 2x = 0$, we have

$$\tan y = \tfrac{1}{2}(\phi + y)^{-1}$$

and

$$x = \phi + \frac{1}{2}\phi^{-1} - \frac{7}{24}\phi^{-3} + \frac{163}{480}\phi^{-5} - \frac{6637}{13440}\phi^{-7} + \cdots. \tag{2}$$

The convergence of this series for values of $n > 1$ is very rapid. Comparison of the second root calculated from trigonometric tables and by the series

using five terms agree to 5 places of decimals, the value of the fifth term in the series being 10^{-5}.

The values of the roots are given below

	$\tan x = -x$	$\tan x = -2x$
x_1	2.02876	1.83660
x_2	4.91318	4.81584
x_3	7.97867	7.91705
x_4	11.08554	11.04083
x_5	14.20744	14.17243
x_6	17.33638	17.30764
x_7	20.46917	20.44480
x_8	23.60428	23.58314
x_9	26.74092	26.72225
x_{10}	29.87859	29.86187
x_{11}	33.01700	

The calculations were carried to 6D and rounded off to 5D. It is not believed that in any case the last figure is in error by more than one unit.

It has been brought to the author's attention by R. P. EDDY, of the Naval Ordnance Laboratory, that in LOTHAR COLLATZ, *Eigenwertprobleme und ihre numerische Behandlung*, Leipzig, 1945, p. 145, are given 4D values of the first 3 roots of $\tan x = -x$, the first 2 roots of $\tan x = \pm 2x$, and the first 4 roots of $\tan x = x$.

It might also be noted that the first 7 roots, 6–10D, of the equations (i) $\cot x + x = 0$, or $J_{-\frac{3}{2}}(x) = 0$, (ii) $\tan x - x = 0$, or $J_{\frac{3}{2}}(x) = 0$, (iii) $\tan x - 3x/(3 - x^2) = 0$, or $J_{\frac{5}{2}}(x) = 0$, (iv) $\tan x + (3 - x^2)/x$, or $J_{-\frac{5}{2}}(x) = 0$, are to be found in NBSMTP, *Tables of Spherical Bessel Functions*, v. 2, 1947, p. 318–319.

L. G. POOLER

U. S. Navy Dept., Bureau of Ordnance

[1] *MTAC*, v. 1, p. 203; see also p. 336, 459 and v. 2, p. 95.—EDITOR.

[2] RAYLEIGH, *Theory of Sound*, second ed., rev. and enl. by R. B. LINDSAY. New York, 1945, v. 1, p. 334.

105. NOTE ON THE FACTORS OF $2^n + 1$.—I have established the primality of

$$N = (2^{92} + 1)/17$$
$$= 29\ 12800\ 09243\ 61888\ 82115\ 58641.$$

This is the fifth largest prime known, the four largest ones being

$2^{127} - 1$	(LUCAS (?) 1876, FAUQUEMBERGUE 1914)
$2^{107} - 1$	(POWERS, Fauquembergue 1914)
$(10^{31} + 1)/11$	(D. H. LEHMER 1927)
$2^{89} - 1$	(Powers 1911, Fauquembergue 1912)

My work is in four steps and is based on the converse of FERMAT's theorem as modified by Lehmer, and may be described briefly as follows.

In step I, the sequence $3, 3^2, 3^4, 3^8, \cdots$ was computed (mod N) by successive squaring. It was found that

$$3^{2^{90}} \equiv -81 \equiv -3^4 \pmod{N}.$$

Hence

$$3 \cdot 3^{2^{92}} = 3^{17N} \equiv 3^{17} \pmod{N}.$$

That is, N "behaves like a prime."

In step II, by combining previously computed values of $3^k \pmod N$ in the appropriate way it was found that $3^{N-1} \equiv 1 \pmod N$.

In step III the following theorem of Lehmer was used. Let p be a prime factor of $N - 1$ and let $N - 1 = mp = hp^a$. Then if N divides $b^{N-1} - 1$ but is prime to $b^m - 1$, all the divisors of N are of the form $p^a x + 1$. Since, in our case

$$N - 1 = 2^4(2^{88} - 1)/17 = 2^4 \cdot 3 \cdot 5 \cdot 23 \cdot 89 \cdot 353 \cdot 397 \cdot 683 \cdot 2113 \cdot 2931542417,$$

the best value of p is 2931542417. It was found that

$$3^m - 1 \equiv 16\ 79443\ 67320\ 76409\ 93695\ 68642 \pmod N,$$

a number prime to N. Hence the theorem applies with $b = 3$ and $a = 1$. The factors of N (if any) are of the form $2931542417x + 1$. It is well known that these factors are also of the form $184x + 1$, and hence of the combined form

$$539403804728x + 1.$$

In step IV the 30 numbers of this form not exceeding the square root of N were examined as possible factors of N. All but 8 of these have prime factors under 100 and hence cannot be primes. None of the 8 others divides N; hence N is a prime.

I have also investigated $N_k = (2^k + 1)/3$ for $k = 67$ and 71. Both are composite since

$$\begin{aligned} &(1) \qquad & 3^{2^{67}} &\equiv 24486\ 86690\ 62763\ 73758 & \pmod{N_{67}} \\ && 3^{2^{71}} &\equiv 6\ 00827\ 62146\ 43042\ 03171 & \pmod{N_{71}} \end{aligned}$$

I discovered the factors of

$$N_{67} = 7327657 \cdot 671\ 31031\ 82899$$

later (see *MTAC*, v. 3, p. 451). The factors of N_{71} are unknown.

A. Ferrier

Collège de Cusset
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Editorial Notes: The four steps of Mr. Ferrier give a valid and positive proof of the primality of $(2^{92} + 1)/17$. However steps II and IV may be shown to be unnecessary. Step IV may be obviated by the following simple reasoning: If N were composite we would have

$$N = (mr + 1)(nr + 1) = A^2 - B^2,$$

where $r = 539403804728$, $m \geqslant n > 0$ and $2A = (m + n)r + 2$. Since the least prime factor of N exceeds r,

$$2A < r + N/r < 10^{14} < r^2.$$

Now

$$N + 1 = 1 + (mr + 1)(nr + 1) \equiv 2 + (m + n)r \equiv 2A \pmod{r^2}.$$

Hence the remainder on division of $N + 1$ by r^2 must be less than 10^{14}. A very rough calculation shows it to be about $3 \cdot 26 \cdot 10^{22}$ however.

The elimination of the more lengthy step II requires a little more reasoning. By the same method of proof the theorem used in step III may be modified to the following: Let p be a prime factor of $N - 1 = mp = hp^a$, and let k be prime to p ($k = 17$ in the above example). Then if N divides $b^{kN} - b^k$ but is prime to $b(b^m - 1)$, all the factors of N are of the form $p^a x + 1$. Thus step II is unnecessary.

Mr. Ferrier's result (1) was obtained also by D. H. L. in October 1946. This comforting agreement, though no longer of much importance in the presence of the factors of N_{67}, adds extra strength to Mr. Ferrier's assertion that N_{71} is composite.

QUERIES

31. Japanese World War II Editions of Mathematical Tables.—From Japanese, and an American colleague serving as an officer with the American Armed Forces in Tokyo, I have learned that offset prints of many mathematical tables were published in Tokyo during this War. One volume of this kind is in the Brown University Library; it is a reproduction of J. Peters, *Sechsstellige Werte der Kreis- und Evolventen-Funktionen von Hundertstel zu Hundertstel des Grades nebst einigen Hilfstafeln für die Zahnradtechnik*. Berlin and Bonn, 1937. viii, 217 p. + 5 blank p. for "Notizen." In the Japanese edition, the Notizen pages are eliminated, the Inhalt (p. iii) occupies the back of the title-page, where former copyright notices were printed, and the print page is slightly smaller and less distinct than the original. Odd-numbered pages occupy the ordinary position of even-numbered pages. The Brown copy bears a Japanese stamp suggesting that it came from the "Educational Institution for Technicians at the Army Fuel Office," and, according to a small label in the volume, it was sold for 5 yen in a Tokyo retail shop specializing in scientific publications.

In one of the Libraries of the University of Tokyo my American colleague saw a Japanese reprint of Bierens de Haan, *Nouvelles Tables d'Intégrales Définies*, 1867, and in the University library catalogue it was indicated as such a reprint. (See *MTAC*, v. 1, p. 321–322.)

Can any reader supply details concerning other tables of this kind, or further facts regarding the volumes mentioned above?

R. C. A.

QUERIES—REPLIES

40. Pitiscus Tables (Q. **29**, v. 3, p. 398).—**A.** The University of Liverpool has copies of the English edition of the Pitiscus *Trigonometrie*, 1630 (*STC* 19968) and of the corresponding *Canon*, 1630 (*STC* 19966). These are bound in a single volume in the Harold Cohen Library (compare *MTAC*, v. 1, p. 170), in a collection of manuscripts, early printed books, and general literature, bequeathed by Thomas Glazebrook Rylands (1818–1900). The printed catalogue of this collection, published in 1900 by the Liverpool University Press, has the following title:

A Catalogue of the Books, printed and in manuscript, bequeathed by the late THOMAS GLAZEBROOK RYLANDS . . . to the Library of University College, Liverpool. Compiled by John Sampson, Librarian to the College. ix, 113 p.

I may add that there is a copy of the 1614 *Canon* (*STC* [19966a]) in the Lincoln Cathedral.

Alan Fletcher

University of Liverpool

B. The Library of the U. S. Naval Observatory in Washington has both the 1630 and 1631 English editions of the Trigonometry of Pitiscus (19968 and 19968a) as well as two copies of the 1630 *Canon* (19966).

In the Library are also **6**, **7**, **14**, **15**, **16** described in *MTAC*, v. 3, p. 391, 394–395.

Mrs. Grace O. Savage
Librarian

C. Editorial Note: Footnote 18, p. 397, (with one correction and the additional location of 18 more library copies), is here reprinted; for information concerning 6 of these copies we are indebted to the Librarian of the Houghton Library, Harvard University: Prof. W. A. Jackson.

[19966a]. *A Canon of Triangles*, [1614], entered in the Stationers' Company Register 17 Jan. 1614. No place of publication, no printer's name, no date. No entry in *STC*; here identified for the first time. Signatures A–L^4, M^2.
Library Copies: British Museum, Brown Univ., Mr. Harrison D. Horblit of New York, Huntington Lib., Lincoln Cathedral, Trinity College, Cambridge.

19966. [Anr. ed.] 1630. 4to. T. Purfoot for J. Tapp, 1630.
Library Copies: Boston Public Lib., British Museum (omitted in *STC*), Univ. Cambridge, Christ Church College Lib., Crawford Lib., Huntington Lib., Univ. Liverpool, Univ. Michigan, U. S. Naval Observatory (2 copies), Yale Univ.

19967. *Trigonometry*. Tr.Ra: Handson, 1614.
Library Copies: Bodleian, British Museum, Mr. Harrison D. Horblit, Huntington Lib., Lincoln Cathedral, Dr. Daniel Williams Lib., London.

19968. [Anr. ed.], 1630.
Library Copies: Boies-Penrose Lib., Boston Public Lib., British Museum (omitted in *STC*), Christ Church College Lib., Huntington Lib., Univ. Liverpool, Pembroke College Lib., U. S. Naval Observatory, Univ. College, London, Yale Univ.

19968a. [Anr. ed.], 1631. Entered in the Stationers' Company Register 1 Aug. 1631.
Library Copies: Univ. Cambridge, Cashell Lib. (Ireland), Crawford Lib., Univ. Illinois, Univ. Michigan, U. S. Naval Observatory. Not in the Huntington Lib. (*STCH*), as stated in *STC*.

CORRIGENDA

V. 1, p. 479, *delete* Weber, H. M. and all page references which follow. Then *add*
Weber, H. F. 72^2, 108, 206, 219, 220, 244, 245, 278, 294, 335, 446
Weber, H. M. 199^2, 293, 303, 307

V. 3, p. 40, l. 6, *delete* Von; p. 314, l. 12, *delete* Pidduck,; p. 361, *for* **593[K]**, *read* **593[I, U]**; p. 385, *for* Lancros, *read* Lanczos; p. 391, l. −4, −5, *for* It is a 7D table for sin, cos; 7–8D table for tan, cot; 8–9D table for sec, csc., *read* The tables, mainly 7D, are for the six trigonometric functions, but they are 7–12D for sin, cos; 5–7D for tan, cot; 5–12D for sec, csc; p. 424, l. 1, *for* $n > 0$, *read* $n \geq 0$.

Symposium Announcement

A second Symposium on large-scale digital calculating machinery is to be held at the Computation Laboratory of Harvard University in eight sessions, September 13 through 16, 1949. The program at present is being prepared and will be announced about July first. It is planned that especial consideration shall be given to the application of computing machinery to the solution of problems in the physical and social sciences. Mark III Calculator will be operating under test conditions before shipment to the Naval Proving Ground, Dahlgren, Virginia.

H. H. Aiken